Convex blocking and partial orders on the plane

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Abstract. Let $C = \{c_1, \ldots, c_n\}$ be a collection of disjoint closed convex sets in the plane. Suppose that one of them, say $c_1$, represents a valuable object we want to uncover. We are allowed to pick a direction $\alpha \in [0, 2\pi)$ along which we can translate (remove) the elements of $C$ one at a time while avoiding collisions. In this paper we solve the problem of finding the direction $\alpha_0$ that minimizes the number of elements of $C$ that have to be removed before we can reach $c_1$ in $O(n^2 \log n)$.

Introduction

Consider a set $C = \{c_1, \ldots, c_n\}$ of pairwise disjoint closed convex sets, and a direction $\alpha \in [0, 2\pi)$; e.g., the vertical upwards direction. It is well known that the elements of $C$ can be translated (removed) one at a time by moving them upwards while avoiding collisions with other elements of $C$ [2, 4]. Suppose that $c_1$ is a special object that we want to uncover, and that we are allowed to choose a direction $\alpha$ along which we can remove the elements of $C$ one at a time while avoiding collisions.

We want to find the direction $\alpha_0$ that minimizes the size of the up-set of $c_1$. For example, in Figure 1(a) it is easy to see that for $\alpha_2$ four elements of $C$ have to be removed before $c_1$ is uncovered, while for $\alpha_1$ we only need to remove two.

This problem can be seen as a variant of the problem known in computational geometry as the “separability problem” [3]. It is also related to spherical orders determined by light obstructions [1].

1 Preliminaries

Given two convex sets $c_i$ and $c_j$ in $C$, we say that $c_j$ is an upper cover of $c_i$ in the $\alpha$ direction (for short, an $\alpha$-cover) if any directed line segment with direction $\alpha$, starting at a point in $c_i$ and ending at a point in $c_j$, does not intersect any other set in $C$. We say that $c_j$ blocks $c_i$ in the direction $\alpha$, written as $c_j \succ_\alpha c_i$, if there is a sequence $c_{\sigma(1)} = c_i, c_{\sigma(2)}, \ldots, c_{\sigma(k)} = c_j$ of elements of $C$ such that $c_{\sigma(i+1)}$ is an $\alpha$-cover of $c_{\sigma(i)}$, $i = 1, \ldots, k-1$.

For each $\alpha$, the blocking relation $\succ_\alpha$ is a partial order on $C$, which is a truncated planar lattice [4]. The planar diagram of such truncated lattice has the elements of $C$
Convex blocking on the plane

(a) A set $C$ of convex sets.

(b) Truncated lattice of $C$ for $\alpha = \pi/2$.

**Figure 1.** Convex sets and its truncated lattice.

as vertices and there is an arc from $c_i$ to $c_j$ if $c_j$ is an $\alpha$-cover of $c_i$ (Figure 1(b)). The elements of $C$ that we need to remove in the $\alpha$ direction before $c_1$ is reached are those convex sets $c_i$ where $c_i \succ_\alpha c_1$, usually defined as the upper set of $c_1$ in $\succ_\alpha$, or *up-set* for short.

**Lemma 1.1** Let $c_i$ and $c_j$ be two convex sets in $C$. The set of directions in which $c_j$ blocks $c_i$ forms a unique non-empty interval $I_{i,j}$. The endpoints of any such $I_{i,j}$ are directions determined by the internal tangents between pairs of elements of $C$.

It follows that there are at most $4\binom{n}{2}$ combinatorially distinct values of $\alpha$ where the truncated lattice changes; these changes occur in slopes parallel to internal tangents between pairs of elements of $C$, in both directions. We can then reduce the search space for $\alpha_0$ to the set $D = \{\gamma_1, \ldots, \gamma_{4\binom{n}{2}}\}$ containing these directions. For the sake of clarity, we are supposing that no two internal tangents are parallel and that the elements of $D$ are ordered as $\gamma_i < \gamma_j$ if $i < j$.

### 2 The transitive triangulation

Our problem can be trivially solved by calculating the truncated lattice for every direction in $D$, obtaining the up-set of $c_1$ in each one, and selecting the $\gamma_i$ with the smallest up-set. Since calculating the truncated lattice has a cost of $O(n \log n)$ time for each of the $4\binom{n}{2}$ directions, this yields an $O(n^3 \log n)$ time algorithm.

To improve this complexity, we calculate the truncated lattice only for the first direction in $D$ instead, and for each $\gamma_i$ we update a triangulation of $\gamma_{i-1}$ in constant time, with $i > 1$.

For each direction $\alpha \in [0, 2\pi)$, we can extend the truncated lattice for $\succ_\alpha$ to a complete lattice by adding two special vertices, a source $s$ and a sink $t$. These vertices will be such that for each maximal element $c_i$ of the relation $\succ_\alpha$, we have $t \succ_\alpha c_i$, and for each minimal $c_j$ we have $c_j \succ_\alpha s$. For a fixed direction we can picture $t$ as a very large convex set standing above all of $C$, and $s$ as a very large convex set standing below all of $C$; (Figure 2(a)).
For each \( \alpha \), we now extend such a complete lattice to a triangulation \( T_\alpha \), which we will call the *transitive triangulation in \( \alpha \)*, by preserving the transitivity of \( \succ_\alpha \), and adding only arcs between convex sets that are visible from each other in the \( \alpha \) direction (Figure 2(b)).

![Complete lattice and transitive triangulation](image)

**Figure 2.** Complete lattice and its corresponding transitive triangulation.

By Lemma 1.1 there are at most \( 4\binom{n}{2} \) lattices, and we want to know how \( T_\alpha \) changes as \( \alpha \) goes from \( \gamma_i \) to \( \gamma_{i+1} \). This leads to the following lemma:

**Lemma 2.1** Given the transitive triangulation \( T_{\gamma_i} \), the transitive triangulation \( T_{\gamma_{i+1}} \) can be obtained by flipping an arc in \( T_{\gamma_i} \). Moreover, such an arc flip either adds or removes an arc between the convex sets \( c_i \) and \( c_j \) that define \( \gamma_{i+1} \).

We omit the proof of Lemma 2.1 because of space restrictions, but an illustration of this fact is shown in Figure 3. We remark that the arc flip can be done preserving transitivity.

### 3 An \( O(n^2 \log n) \) algorithm to find \( \alpha_0 \)

**Theorem 3.1** Finding \( \alpha_0 \) can be done in \( O(n^2 \log n) \).

We need the following results, given without proof:

**Lemma 3.2** The up-set of \( c_1 \) changes in \( T_{\gamma_1}, \ldots, T_{\gamma_4(2)} \) at most a linear number of times.

**Lemma 3.3** Given \( T_{\gamma_i} \) we can answer the query of whether \( c_j \) is in the up-set of \( c_k \) in \( O(\log n) \) time.

We observe that \( D \) can be calculated in \( O(n^2 \log n) \) if we suppose that the internal tangents between any two convex sets in \( C \) can be determined in constant time. For each \( \gamma_i \) we store the indexes \( j, k \) of the convex sets that define the internal tangent. There are three types of events involving that can arise when we process \( \gamma_i \):

1. \( c_k \) and \( c_j \) are not in the up-set of \( c_1 \). In this case the up-set of \( c_1 \) remains unchanged.
Convex blocking on the plane

Figure 3. An example of an arc flip: The arc $c_a \rightarrow c_b$ replaces $c_i \rightarrow c_j$ as $c_j$ no longer blocks $c_i$ in the $\gamma_{i+1}$ direction. Observe that $c_a \rightarrow c_b$ preserves transitivity.

(2) $c_k$ and $c_j$ belong to the up-set of $c_1$. If $c_k$ and $c_j$ become comparable, the up-set of $c_1$ remains. Suppose that $c_k$ and $c_j$ become un-comparable. It can be proved that by checking a constant number of the neighbors of each of $c_k$ and $c_j$, we can verify whether they remain in the up-set of $c_1$. If they do, then the up-set of $c_1$ remains. If not, then we recalculate in $O(n)$ time the up-set of $c_1$.

(3) A similar process happens when exactly one of $c_k$ and $c_j$ belongs to the up-set of $c_1$.

By Lemma 3.2, we have to update the up-set of $c_1$ only a linear number of times, and thus the whole process takes $O(n^2 \log n)$ time. This proves Theorem 3.1.

References


