Provability logics and proof-theoretic ordinals

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Provability in modal logic

Modal language:

\[
p, \neg \varphi, \varphi \land \psi, \Box \varphi
\]

Arithmetic interpretation: assigns a formula \( p^* \) in the language of arithmetic to each propositional variable \( p \).

- \( p \mapsto p^* \)
- \( * \) commutes with Booleans
- \( (\Box \varphi)^* = \text{Prov}_T(\neg \varphi^*) \)

Validity: The formula \( \varphi \) is valid (with respect to \( T \)) if, for every arithmetic interpretation \( * \), \( T \vdash \varphi^* \).

Second incompleteness theorem:

\[
\Box \Diamond T \rightarrow \Box \bot
\]
Gödel-Löb logic

Axioms:

- All tautologies and modus ponens.
- □(ϕ → ψ) → (□ϕ → □ψ)
- □(□ϕ → ϕ) → □ϕ (Löb’s axiom)
- □ϕ

Theorem (Solovay)

Let T be a sound extension of PA. Then, GL ⊢ ϕ if and only if ϕ is valid.

- De Jongh et. al.: Peano Arithmetic can be replaced with Elementary Arithmetic:
  
  \[ EA = \text{I} \Delta_0 + \text{exp} \]

- We may work in any language capable of coding arithmetic: second-order arithmetic, set theory, etc.
Polymodal Gödel-Löb

GLP$_\Lambda$: $\Lambda$ is a linear order, one modality $[\xi]$ for each $\xi \in \Lambda$.

Axioms:

$$[\xi](\varphi \to \psi) \to ([\xi] \varphi \to [\xi] \psi) \quad (\xi \in \Lambda)$$

$$[\xi]([\xi] \varphi \to \varphi) \to [\xi] \varphi \quad (\xi \in \Lambda)$$

$$[\xi] \varphi \to [\zeta] \varphi \quad (\xi < \zeta \in \Lambda)$$

$$\langle \xi \rangle \varphi \to [\zeta] \langle \xi \rangle \varphi \quad (\xi < \zeta \in \Lambda)$$

Introduced by Japaridze in 1988 with $\Lambda = \omega$.

(Possible) arithmetic interpretations of GLP$_\omega$:

- $[n] \varphi \equiv \text{“}\varphi\text{ is provable from the set of true }\Pi_n^0\text{ sentences”}$.  
- $[n] \varphi \equiv \text{“}\varphi\text{ is provable using }\omega\text{-rules of depth at most }n\text{”}$.  

Worms and ordinals below $\varepsilon_0$

**Worms:** Iterated consistency statements

$$\langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_m \rangle \top$$

**Abbreviations:**

- $123 := \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \top$  \hspace{1cm} (worms as strings)
- $(12)0(4) := 1204$  \hspace{1cm} (zero-concatenation)
- $1 \uparrow 021 := 132$  \hspace{1cm} (one-promotion)

**Order-types:** Worms are well-ordered by their consistency strength:

$$w < v \iff \text{GLP} \vdash w \rightarrow \langle 0 \rangle v.$$  

- $o(\top) = 0$
- $o(w0v) = o(v) + 1 + o(w)$
- $o(1 \uparrow w) = \omega^{o(w)}$
Beklemishev’s analysis of Peano Arithmetic

Turing progressions: Given a theory $T$, define

1. $T_0 = T$
2. $T_{\xi+1} = T_{\xi} + \text{Con}(T_{\xi})$
3. $T_\lambda = \bigcup_{\xi<\lambda} T_{\xi}$

$\Pi_1^0$-ordinal of a theory $T$ over a base theory $U$:

$$\|T\|_{\Pi_1^0}^U = \sup\{\xi : T \vdash \text{Con}(U_{\xi})\}$$

Theorem: $\text{PA} \equiv \text{EA} + \{\langle n \rangle_{\text{EA}} T : n \in \mathbb{N}\}$.

Theorem: For any worm $w$, $T + w_T \equiv_{\Pi_1^0} T_{o(w)}$.

Theorem: $\|\text{PA}\|_{\Pi_1^0}^{\text{EA}} = \varepsilon_0$
Transfinite Gödel-Löb

Beklemishev considered GLP\(_\Lambda\) with \(\Lambda\) an arbitrary ordinal.

Transfinite worms recursively:

- \(\top\)
- \(\omega 0v\)
- \(\lambda \uparrow v\): \(\langle \alpha \rangle \mapsto \langle \lambda + \alpha \rangle\) (w.l.o.g. \(\lambda = \omega^\rho\)).

Order-type calculus:

- \(o(\top) = 0\)
- \(o(\omega 0v) = o(v) + 1 + o(v)\)
- \(o(\omega^\rho \uparrow v) = \varphi^\rho(-1 + o(v))\)

Beklemishev’s autonomous notations: Write \(\langle v \rangle\) instead of \(\langle o(v) \rangle\).

Gives an ordinal notation system for \(\Gamma_0\).
Beklemishev’s autonomous worm notation

1  ()
ω  (())
ε₀  (((())))

2  (())
ω + ω  ((())()(()))
ω^{ε₀+1}  (((()())())())
Systems of second-order arithmetic

**Goal:** Use transfinite provability logic for ordinal analysis up to $\text{ATR}_0$.

- **RCA$_0$**: Second-order arithmetic with $\Delta^0_1$ comprehension and $\Sigma^0_1$ induction.

- **ACA$_0$**: RCA$_0$ with comprehension for all arithmetic formulas.

- **ATR$_0$**: Define

  $$\text{TR}^\Lambda_{\phi}(X, Y) \equiv \forall n\forall \lambda (n \in Y \iff \phi(n, \lambda, Y_{<\lambda}, X))$$

  ATR$_0$ is ACA$_0$ with the axiom scheme

  $$\forall X\forall \Lambda \left( \omega \circ (\Lambda) \rightarrow \exists Y \text{TR}^\Lambda_{\phi}(X, Y) \right).$$

  The formula $\phi$ is arithmetic, possibly with set parameters.
The $\omega$-rule interpretation

Within second-order arithmetic, we may interpret $[\lambda]_T \phi$ using transfinitely iterated $\omega$-rules:

- $[0]_T \phi \iff \Box_T \phi$

- if $\xi < \lambda$,

\[
\frac{[\xi]_T \psi(\bar{0}) \ [\xi]_T \psi(\bar{1}) \ [\xi]_T \psi(\bar{2}) \ldots \ \forall n \psi(n)}{\Box_T (\forall n \psi(n) \rightarrow \phi)} [\lambda]_T \phi
\]

- $[\lambda|X]_T \phi$ means that we may also use an oracle for $X$:

\[
n \in X \iff [0|X]_T (\bar{n} \in \bar{X}) \quad n \notin X \iff [0|X]_T (\bar{n} \notin \bar{X})
\]
Towards a \( \Pi^0_1 \) analysis of predicativity

Predicative consistency:

\[
\text{PredCon}(\mathcal{T}) = \forall \forall \mathcal{X} (\text{wo}(\Lambda) \rightarrow \langle \Lambda | X \rangle_{\mathcal{T}} \top)
\]

Theorem (Cordón-Franco, DFD, JjJ, Lara-Martín)

\[
\text{ATR}_0 \equiv \text{RCA}_0 + \text{PredCon} (\text{RCA}_0)
\]

Conjectures:

- \( \text{ATR}_0 \equiv \Pi^0_1 \text{EA} + \{ \langle \gamma \rangle_{\text{EA}} \top : \gamma < \Gamma_0 \} \)
- \( \| \text{ATR}_0 \|_{\Pi^0_1}^{\text{RCA}_0} = \Gamma_0 \)
Beyond predicativity

**Π₁¹-CA**: Add to RCA₀ all axioms of the form

$$\forall X \exists Y \forall n \ (n \in Y \iff \forall Z \ \phi(n, X, Z))$$

where $\phi$ is arithmetic.

**Impredicativity**: The set $Y$ is defined using a collection which includes $Y$!

$[\infty]_T \phi$ holds if $\phi$ is provable using an arbitrary number of $\omega$-rules.

**Conjecture**:

$$\Pi_1^1\text{-CA} = \text{RCA}_0 + \forall X \langle \infty | X \rangle_{\text{RCA}_0} \top$$
Non-recursive ordinals

To analyze impredicative systems of arithmetic we need to represent non-recursive ordinals.

$\omega_1^{CK}$: First non-recursive ordinal.

**Fact:** The $\omega$-rule becomes saturated in $\omega_1^{CK}$ steps. Thus:

- $[\omega_1^{CK}]\phi \equiv [\omega_1^{CK} + 1]\phi$
- The scheme $\langle \omega_1^{CK} \rangle \phi \rightarrow [\omega_1^{CK} + 1] \langle \omega_1^{CK} \rangle \phi$ fails.

**Problem:** How do we represent non-recursive ordinals in provability algebras?

**Solution:** Use rules that are stronger than the $\omega$-rule.
Impredicative provability logic

\( \omega_1^{CK} \)-rule:

\[
\begin{array}{c}
\langle \psi(\xi) : \xi < \omega_1^{CK} \rangle \\
(\forall x < \omega_1^{CK} \psi(x)) \implies \phi
\end{array}
\]

\( [\xi] T \phi \): “The formula \( \phi \) is provable using \( \xi \) iterations of the \( \omega_1^{CK} \)-rule”.

\( [\xi] T \phi \) holds if either

- \( [\zeta] T \phi \) holds for some recursive \( \zeta \) or
- there are \( \psi \) and \( \zeta < \xi \) such that

\[
(\forall \delta < \omega_1^{CK} [\zeta] T \psi(\delta) \quad \text{and} \quad [0] T ((\forall x < \omega_1^{CK} \psi(x)) \implies \phi)).
\]
Two-typed notations for $\psi(\Gamma_{\omega_1^{CK}+1})$

Expressions of the form $[\xi]$ require $\xi$ to be recursive. Thus, when given a non-recursive argument, it must be **collapsed** to a recursive one.

Notations are built by two types of parentheses: () and ( )

We reduce to ordinary worms in $\text{GLP}_{\omega_2^{CK}}$:

- $(\xi) \mapsto \langle \xi \rangle$ \hspace{1cm} ($\xi < \omega_1^{CK}$)
- $(\omega_1^{CK} + \xi) \mapsto \langle \psi(\xi) \rangle$
- $(\xi) \mapsto \langle \omega_1^{CK} + \xi \rangle$

**Note:** We include the function $\varphi$ when collapsing.
Some impredicative worms

| 1  | (()) | \(\omega\) | (()) |
| \(\omega_1^{CK}\) | (()) | \(\Gamma_0\) | (()) |
| \(\omega_1^{CK} + 1\) | (())(()) | \(\varepsilon_{\omega_1^{CK} + 1}\) | (((())))(()) |
| \(\psi(\varepsilon_{\omega_1^{CK} + 1})\) | (((()())(())) | \(\varphi_{\omega_1^{CK}}(\omega_1^{CK})\) | (())(()) |

But... \(\|\Pi_1^1\text{-CA}\|_{\Pi_0^1}^{RCA_0}\) has to be much bigger than \(\psi(\Gamma_{\omega_1^{CK} + 1})\) ...
The ordinals $\omega^\xi_{CK}$:

- $\omega^\xi_{CK+1}$ is the least ordinal $\kappa$ such that there is no unbounded recursive function $f : \omega^\xi_{CK} \rightarrow \kappa$.

- $\omega^\xi_{CK} = \bigcup_{\zeta < \lambda} \omega^\zeta_{CK}$ for $\lambda \in \text{Lim}$.

$\omega^\xi_{CK}$-rule:

\[
\left\langle \phi(\delta) : \delta < \omega^\xi_{CK} \right\rangle \quad (\forall x < \omega^\xi_{CK} \psi(x)) \rightarrow \phi
\]

$\left[^{\alpha}_\beta\right]_T \phi : \text{“The formula } \phi \text{ is provable over } T \text{ using } \alpha \text{ iterations of the } \omega^\beta_{CK} \text{-rule.”}$
Spiders

- A natural extension of Beklemishev’s autonomous worm notations.

- In $\left[ \frac{\alpha}{\xi} \right] T \phi$ we require $\alpha < \omega_{\xi+1}^{CK}$, so we now collapse $\omega_{\xi+1}^{CK} + \alpha \mapsto \psi_{\omega_{\xi+1}^{CK}}(\alpha)$.

$$
\begin{align*}
\omega & \quad \left( \begin{array}{c} \\
\end{array} \right) & \varphi_{\omega_1^{CK}}(1) & \left( \begin{array}{c} 0 \\
0
\end{array} \right) \\
\omega_1^{CK} & \quad \left( \begin{array}{c} \\
\end{array} \right) & \omega_3^{CK} + \omega_1^{CK} & \left( \begin{array}{c} 0 \\
0 0 0 0
\end{array} \right) \\
\omega_\omega^{CK} & \quad \left( \begin{array}{c} 0 \\
\end{array} \right) & \psi_{\omega_1^{CK}}(\omega_\omega^{CK}) & \left( \begin{array}{c} \left( \begin{array}{c} 0 \\
\end{array} \right) \\
\end{array} \right) \\
\omega_{\omega_1^{CK}}^{CK} & \quad \left( \begin{array}{c} \\
\end{array} \right) & \psi_{\omega_2^{CK}}(\omega_{\omega_1^{CK}}^{CK}) & \left( \begin{array}{c} \left( \begin{array}{c} 0 \\
\end{array} \right) \\
0
\end{array} \right)
\end{align*}
$$
Computing $\|\Pi_1^1 - \text{CA}\|_{\Pi_1^0}^{\text{RCA}_0}$

Conjecture:

$$\Pi_1^1 - \text{CA} \equiv_{\Pi_1^0} \text{RCA}_0 + \forall n < \omega \langle 0 \rangle_{\text{RCA}_0}^n \top$$

Conjecture:

$$\|\Pi_1^1 - \text{CA}\|_{\Pi_1^0}^{\text{RCA}_0} = \psi(\omega^\text{CK}_\omega)$$
Thank you!