On hyper-arithmetic reflection principles

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In Memoriam: Grisha Mints
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- $T_\lambda := \bigcup_{\alpha < \lambda} T_\alpha$ for limit $\lambda < \Gamma$. 
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Schmerl (1978): reflexive transfinite induction
Transfinite induction: $\forall \alpha (\forall \beta < \alpha \phi(\beta) \rightarrow \phi(\alpha)) \rightarrow \forall \alpha \phi(\alpha)$;
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Theorem EA proves reflexive transfinite induction (Schmerl)

If $EA \vdash \forall \alpha \left( \Box_{EA} \ \forall \beta < \dot{\alpha} \ \phi(\beta) \rightarrow \phi(\alpha) \right)$, then

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The dual notion “consistent with $T$ and all true $\Pi_n$ sentences” is denoted $\langle n \rangle_T$. 
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Transfinite progressions are not expressible in the modal language with just one modal operator.

However:

**Proposition:** \(T + \langle 1 \rangle_T \top\) is a \(\Pi_1\) conservative extension of \(T + \{\langle 0 \rangle^k_T \top \mid k \in \omega\}\).
Definition

The logic GLP_\Lambda is the propositional normal modal logic that has for each \( \xi < \Lambda \) a modality \([\xi]\) and is axiomatized by the following schemata:

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\begin{align*}
[\xi](A \rightarrow B) & \rightarrow ([\xi]A \rightarrow [\xi]B) \\
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A & \rightarrow [\xi]A
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\langle \xi \rangle A & \to [\xi]\langle \xi \rangle A & \text{for } \xi < \zeta, \\
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The rules of inference are Modus Ponens and necessitation for each modality: $\frac{\psi}{[\zeta]\psi}$. 
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We can define natural orderings \(<_\xi\) on \(\mathbb{W}\) by

\[ A <_\xi B \iff \text{GLP} \vdash B \rightarrow \langle \xi \rangle A \]
\[ A \prec_\xi B :\iff \text{GLP} \vdash B \rightarrow \langle \xi \rangle A \]

- For \( \prec_0 \) defines a well-order on the class of worms modulo provable GLP equivalence.
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- For \( \prec_\xi \) with \( \xi > 0 \) the relation is no longer linear (mod prov. equivalence) but is still well-founded

By \( o^*(A) \) we denote the order type of \( A \) under \( \prec_0 \) and we write \( o(A) \) instead of \( o^*(A) \).
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- **Definition** By \( o_\alpha(A) \) we denote the order type of \( A \) under \( \prec_\alpha \) and we write \( o(A) \) instead of \( o_0(A) \).
- Worms of \( \text{GLP}_\omega \) are known to be useful for Turing progressions:
- **Proposition** (Beklemishev) For each ordinal \( \alpha < \varepsilon_0 \) there is some \( \text{GLP}_\omega \)-worm \( A \) such that \( o(A) = \alpha \), and \( T + A \) is \( \Pi_1 \) equivalent to \( T_\alpha \).
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For two Ignatiev sequences $\vec{A}$ and $\vec{B}$ we define an accessibility relation $<_n$:
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For two Ignatiev sequences $\tilde{A}$ and $\tilde{B}$ we define an accessibility relation $<_n$:

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We define $\models$ by $\vec{A} \models \top$, for no $\vec{A}$, $\vec{A} \models \bot$.

Theorem GLP$_0$$_\omega$ $\vdash \phi$ $\iff$ $\mathcal{I} \models \phi$
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$\models$ commutes with Boolean connectives: $\vec{A} \models \phi \land \psi$ if and only if $\vec{A} \models \phi$ and $\vec{A} \models \psi$, etc
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$\vec{A} \models \langle n \rangle \phi$ if and only if there is some $\vec{B}$ with $\vec{A} > n \vec{B}$ so that $\vec{B} \models \phi$. 

Theorem GLP$_0$ $\omega \vdash \phi$ $\iff$ $\mathcal{I} \vdash \phi$

Proof by a p-morphic embedding of this structure into the generalization of Ignatiev’s model.

Let’s see a picture.
Let $\mathcal{I}$ denote the set of all Ignatiev sequences.

We define a Kripke frame:

$$\langle \mathcal{I}, \{>n\}_{n \in \omega} \rangle$$

We shall denote this frame also by $\mathcal{I}$.

We define $\models$ by $\bar{A} \models \top$, for no $\bar{A}$, $\bar{A} \models \bot$.

$\models$ commutes with Boolean connectives: $\bar{A} \models \phi \land \psi$ if and only if $\bar{A} \models \phi$ and $\bar{A} \models \psi$, etc.

$\bar{A} \models \langle n \rangle \phi$ if and only if there is some $\bar{B}$ with $\bar{A} >_n \bar{B}$ so that $\bar{B} \models \phi$.

**Theorem** $\text{GLP}_\omega^0 \models \phi \iff \mathcal{I} \models \phi$.
Let $\mathcal{I}$ denote the set of all Ignatiev sequences

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**Theorem** $\text{GLP}_0^0 \vdash \phi \iff \mathcal{I} \vDash \phi$

**Proof** by a p-morphic embedding of this structure into the generalization of Ignatiev’s model.
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**Theorem** $\text{GLP}_0^\omega \vdash \phi \iff \mathcal{I} \vdash \phi$

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Let’s see a picture.
We define the $\Pi_{n+1}$ proof-theoretic ordinal of a theory $U$ as follows:
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$\text{tt}(U) := \bigcup_{n=0}^{\infty} T_{|U|_{\Pi_{n+1}}}^n$
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$\text{tt}(U) := \bigcup_{n=0}^\infty T^n_{|U|_{\Pi_{n+1}}}$

In case $U \equiv \text{tt}(U)$ we say that $U$ has a convergent *Turing-Taylor* expansion.
We will now link Ignatiev’s model to Turing-Taylor expansions
▶ We will now link Ignatiev’s model to Turing-Taylor expansions
▶ Let us recall:

\[
\begin{align*}
T_0 & := T; \\
T_{\alpha + 1} & := T_\alpha \cup \{ \langle i \rangle \}_{T_\alpha} \top; \\
T_\lambda & := \bigcup_{\alpha < \lambda} T_\alpha \text{ for limit } \lambda.
\end{align*}
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- $T_0^i := T$
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We shall use the ordinal notation system $\langle B_n, <n \rangle$ to label the Turing progression based on $n$-consistency.
We will now link Ignatiev’s model to Turing-Taylor expansions

Let us recall:

1. $T_i^0 := T$
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Thus, $T_3^1$ denotes $T_ω^1$.
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- $T^i_0 := T$;
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We shall use the ordinal notation system $\langle B_n, <_n \rangle$ to label the Turing progression based on $n$-consistency.

Thus, $T^1_3$ denotes $T^1_{\omega \omega}$, and $T^2_3$ denotes $T^2_{\omega}$.
We will now link Ignatiev’s model to Turing-Taylor expansions.

Let us recall:

- \( T_0^i := T \);
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- \( T_\lambda := \bigcup_{\alpha < \lambda} T_\alpha \) for limit \( \lambda \).

We shall use the ordinal notation system \( \langle B_n, <_n \rangle \) to label the Turing progression based on \( n \)-consistency.

Thus, \( T_3^1 \) denotes \( T^1_{\omega \omega} \),

and \( T_3^2 \) denotes \( T^2_\omega \),

and \( T_3^3 \) denotes \( T^2_1 \).
Theorem

For each worm $A$:

$$T + A \equiv \bigcup_{n=0}^{\infty} T^n(A)$$
Theorem
For each worm $A : T + A \equiv \bigcup_{n=0}^{\infty} T_{h_n}(A)$

Theorem
For each worm $A : T + A \equiv \bigcup_{n=0}^{\infty} T_{A}^{n}$
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For each worm $A$: $T + A \equiv \bigcup_{n=0}^{\infty} T_A^n$

Compare this to

$$f(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
Theorem The Ignatiev sequences exactly correspond to those sub-theories of PA that have a convergent Turing-Taylor expansion.
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and for each $\tilde{A} \in \mathcal{I}$, there is a theory $U$ so that $tt(U) = \tilde{A}$
Theorem The Ignatiev sequences exactly correspond to those sub-theories of $\mathbb{PA}$ that have a convergent Turing-Taylor expansion

That is, for each such theory $U$, we have that $\text{tt}(U) \in I$

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This yields a roadmap to conservation results!
We proof of the theorem uses three main results.
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We shall see why worms are better than the more familiar ordinal notations in this context
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For each number $n$ and each GLP$_{\omega}$ worm $A$, $\text{GLP}_\omega \vdash A \leftrightarrow h_n(A) \land r_n(A)$
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For each number $n$ and each $\text{GLP}_\omega$ worm $A$, $\text{GLP}_\omega \vdash A \leftrightarrow h_n(A) \land r_n(A)$ (here, $r_n(A)$ denotes the $n$ remainder of $A$ so that $A = h_n(A)r_n(A)$)
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- For each worm $A \in W_n$ we have $T + A \equiv_n T^n_A$ (Beklemishev)
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For each worm $A$:
$$T + A \equiv \bigcup_{n=0}^{\infty} T_{h_n(A)}^n$$ (JjJ)
We proof of the theorem uses three main results

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- For each worm $A : T + A \equiv \bigcup_{n=0}^{\infty} T^n_{h_n(A)}$ (JjJ)

Corollaries:
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  - For each number \( n \) and each GLP_\( \omega \) worm \( A \),
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  - For each worm \( A \in \mathbb{W}_n \) we have \( T + A \equiv_n T^n_A \) (Beklemishev)
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Corollaries:

- For each worm $A \in W_n$ we have $T + nA \equiv T^n_{nA}$
- For each worm $A \in W_n$ $T^n_A \vdash T^m_A$ for $m < n$
tt(U) denotes \langle |U|_{\Pi_1^0}, |U|_{\Pi_2^0}, |U|_{\Pi_3^0}, \ldots \rangle.
tt(\(U\)) denotes \(\langle |U|_{\Pi_1^0}, |U|_{\Pi_2^0}, |U|_{\Pi_3^0}, \ldots \rangle\).

Likewise, with every sequence \(\vec{\alpha} = \langle \alpha_0, \alpha_1, \ldots \rangle\) of ordinals below \(\varepsilon_0\) we can naturally associate a sub theory \((\vec{\alpha})_{tt}\) of \(\text{PA}\) as follows

\[
(\vec{\alpha})_{tt} := \bigcup_{n=0}^{\infty} \text{EA}^n_{\alpha_n}.
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tt(\mathcal{U}) \text{ denotes } \langle |\mathcal{U}|_{\Pi^0_1}, |\mathcal{U}|_{\Pi^0_2}, |\mathcal{U}|_{\Pi^0_3}, \ldots \rangle.

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Likewise, with every sequence \vec{A} = \langle A_0, A_1, \ldots \rangle of GLP_\omega worms we can naturally associate a sub theory \langle \vec{A} \rangle_{tt} of PA as follows

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The monomials in Turing-Taylor progressions are the $T^n_A$.
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Example: $T_1^1 + T_0^0 \equiv T_1^1 + T_0^0$.
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The monomials in Turing-Taylor progressions are the $T^n_A$.

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Example: $T_1^1 + T_0^0 \equiv T_1^1 + T_{101}^0$.

$T_1^1 \equiv T + \langle 1 \rangle \top$, and

$T_0^0 \equiv T_{01}^0$. 

Thus, $T_1^1 + T_0^0 \equiv T + \langle 1 \rangle \top + \langle 0 \rangle \langle 1 \rangle \top$. 

$\equiv T_1^1 + T_{0\omega}^0 + 2$.

In the classical notation system this reads $T_1^1 + T_{0\omega}^{\omega+1} \equiv T_1^1 + T_{0\omega}^{\omega \cdot 2}$.

The example shows that in general $\text{tt}((\vec{A})_{\text{tt}}) \neq \vec{A}$!
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With a slightly more involved reasoning, we can prove...
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Using these ingredients one easily proves
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Theorem If $U$ is some sub-theory of PA with a convergent Turing-Taylor expansion, so that $U \not\equiv_0 PA$, then $\text{tt}(U)$ defines a point in $\mathcal{I}$. 

Joost J. Joosten

On hyper-arithmetic reflection principles
We would like to extend the results of the first section beyond first order
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Friedman, Godlfarb and Harrington come to the rescue!
Definition (Witness-comparison relation)

For $\phi := \exists x \phi_0(x)$ and $\psi := \exists x \psi_0(x)$ we define $\phi \leq \psi := \exists x (\phi_0(x) \land \forall y < x \neg \psi_0(x))$ and $\phi < \psi := \exists x (\phi_0(x) \land \forall y \leq x \neg \psi_0(x))$.

Theorem (Rosser's Theorem)

Let $T$ be a consistent c.e. theory extending $\text{EA}$. There is some $\rho \in \Sigma_0^1$ which is undecidable in $T$. That is, $T \nvdash \rho$ and $T \nvdash \neg \rho$.

Proof

Consider $\rho \leftrightarrow \neg (2^\rho < 2^{\neg \rho})$.

I find it utterly amazing that something sensible can be proven using the witness comparison techniques!

Joost J. Joosten On hyper-arithmetic reflection principles
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Theorem (Rosser’s Theorem)

Let $T$ be a consistent c.e. theory extending $\text{EA}$. There is some $\rho \in \Sigma^0_1$ which is undecidable in $T$. That is,

$$T \not\vdash \rho \quad \text{and},$$

$$T \not\vdash \neg \rho.$$

Proof Consider $\rho \leftrightarrow \neg (\Box \rho < \Box \neg \rho).$
Definition (Witness-comparison relation)

For $\phi := \exists x \phi_0(x)$ and $\psi := \exists x \psi_0(x)$ we define

$$\phi \leq \psi := \exists x (\phi_0(x) \land \forall y < x \neg \psi_0(x))$$

and,

$$\phi < \psi := \exists x (\phi_0(x) \land \forall y \leq x \neg \psi_0(x)).$$

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$$T \nvdash \rho \quad \text{and,}$$

$$T \nvdash \neg \rho.$$
Lemma

Let $A$ and $B$ be some formulas of logical complexity $\Sigma_{n+1}^0$ for $n < \omega$.

1. Both $A < B$ and $A \leq B$ are of complexity $\Sigma_{n+1}^0$;
2. $EA \vdash (A < B) \rightarrow (A \leq B)$;
3. $EA \vdash (A < B) \rightarrow (A \leq B)$;
4. $EA \vdash (A \leq B) \rightarrow A$;
5. $EA \vdash (A \leq B) \rightarrow \neg(B < A)$ and consequently;
6. $EA \vdash (A < B) \rightarrow \neg(B \leq A)$;
7. $EA \vdash [(B \leq B) \lor (A \leq A)] \rightarrow [(A \leq B) \lor (B < A)]$;
8. $EA \vdash A \land \neg(A \leq B) \rightarrow B$. 

Theorem (FGH theorem)

Let $T$ be any computably enumerable theory extending $\mathsf{EA}$. For each $\sigma \in \Sigma^0_1$ we have that there is some $\rho \in \Sigma^0_1$ so that $\mathsf{EA} \vdash (\sigma \leftrightarrow 2^T \rho)$.

Proof.
Consider the fixpoint $\rho$ for which $\mathsf{EA} \vdash \rho \leftrightarrow (\sigma \leq 2^T \rho)$.

This shows us that we can express a syntactical class using provability logics!

We wish to stretch this further.
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Let $T$ be any computably enumerable theory extending $\text{EA}$. For each $\sigma \in \Sigma^0_1$ we have that there is some $\rho \in \Sigma^0_1$ so that

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**Lemma**

*Let $A \in \Sigma_{n+1}^0$, then the schema $A \rightarrow (A \leq A)$ is over EA provably equivalent to the least-number principle for $\Delta_n^0$ formulas.*
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- so FGH is generalizable over a weak base theory.
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**proposition**

Let $T$ be a computable theory extending EA and let $\phi$ be a $\Sigma^0_{n+1}$ formula. We have that

$$EA \vdash \phi \rightarrow [n]^\text{True}_T \phi.$$
Theorem

Let $T$ be any computably enumerable theory extending $\text{EA}$ and let $n < \omega$. For each $\sigma \in \Sigma_{n+1}^0$ we have that there is some $\rho_n \in \Sigma_{n+1}^0$ so that

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▶ proof The proof runs entirely analogue to the proof of the classical FGH theorem.
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▶ Thus, for each number $n$ we consider the fixpoint $\rho_n$ so that

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▶ Corollary

Let \( T \) be a c.e. theory extending \( \text{EA} \) and let \( n \in \mathbb{N} \). For each formulas \( \varphi, \psi \) there is some \( \sigma \in \Sigma_{n+1}^0 \) so that

\[
T \vdash ([n]^\text{True}_T \varphi \lor [n]^\text{True}_T \psi) \leftrightarrow [n]^\text{True}_T \sigma.
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The \([n]^{\text{True}}\) predicates tie up with the arithmetical hierarchy:
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Let $T$ be any c.e. theory and let $A \subseteq \mathbb{N}$. The following are equivalent:

1. $A$ is c.e. in $\emptyset^{(n)}$;
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3. $A$ is definable on the standard model by a $\Sigma_0^{n+1}$ formula;
4. $A$ is definable on the standard model by a formula of the form $[n]^{\text{True}}_T \rho(\dot{x})$;
5. $A$ is definable on the standard model by a formula of the form $[n]^{\text{True}}_T \rho(\dot{x})$ where $\rho(x) \in \Sigma_0^{n+1}$.  

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On hyper-arithmetic reflection principles
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\[
[n + 1]_T^{\Omega} \varphi := \exists \psi \left( \forall x [n]_T^{\Omega} \psi(x) \land \Box_T (\forall x \psi(x) \rightarrow \varphi) \right)
\]

\[
[n]_T^{\Omega} \text{ is a } \Sigma^0_{2n+1} \text{-formula.}
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**Lemma**

*Let $T$ be a computable theory extending $\mathsf{EA}$ and let $\phi$ be a $\Sigma^0_{2n+1}$ formula. We have that*

\[
\mathsf{EA} \vdash \phi \rightarrow [n]_T^{\Omega} \phi.
\]

**Proof.**

By an external induction on $n$ where each inductive step requires the application of an additional omega-rule.
Corollary

Let $T$ be any computably enumerable theory extending $\mathbb{EA}$ and let $n < \omega$. For each $\sigma \in \Sigma^0_{2n+1}$ we have that there is some $\rho_n \in \Sigma^0_{2n+1}$ so that

$$\mathbb{EA} \vdash \langle n \rangle^\Omega_T \top \rightarrow (\sigma \leftrightarrow [n]^\Omega_T \rho_n).$$
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Let $T$ be any c.e. theory, let $n$ be a natural number, and let $A \subseteq \mathbb{N}$. The following are equivalent:

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Runs out of phase!

We wish to use the best of both worlds

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We wish to use the best of both worlds
\[ [0]^T \phi := \Box T \phi, \quad \text{and} \]
\[ [n + 1]^T \phi := \Box T \phi \lor \exists \psi \bigg( \bigwedge_{0 \leq m \leq n} \left( \langle m \rangle^T \psi \land \Box \langle m \rangle^T \psi \rightarrow \phi \right) \bigg). \]
Let $T$ be a sound c.e. theory extending $\text{EA}$. We have for all $n \in \mathbb{N}$ that

1. $\text{EA} \vdash \forall \varphi \ (([n]^T_\square \varphi \rightarrow [n]^T_{\text{True}} \varphi))$;
2. $\text{EA} \vdash \langle n \rangle^T_{\text{True}} \top \rightarrow \forall \varphi \ ([n + 1]^T_\square \varphi \leftrightarrow [n + 1]^T_{\text{True}} \varphi))$;
3. $\text{EA} \vdash [n]^T_{\text{True}} \left( \forall \varphi \ ([n]^T_\square \varphi \leftrightarrow [n]^T_{\text{True}} \varphi) \right)$;
4. $\mathbb{N} \models \forall \varphi \ ([n]^T_\square \varphi \leftrightarrow [n]^T_{\text{True}} \varphi)$. 
Theorem

Let $T$ be a c.e. theory. We have for all $A \subseteq \mathbb{N}$ that the following are equivalent

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Theorem

The logic $\text{GLP}_\Lambda$ is sound for strong enough theories $T$ under the interpretation $\Box \mapsto [\lambda]^\Box,\Lambda_T$. 
Definition
Let $T$ be a c.e. theory. We define

- $\Delta^0_0 := \Sigma^0_0 := \Pi^0_0 := \Delta^0_0$;
- $\Sigma^\alpha_{\alpha+1} = \Sigma^\alpha_\alpha \cup \Pi^\alpha_\alpha \cup \{[\alpha]_T^\square \varphi(\dot{x}) \mid \varphi(x) \in \text{Form}\}$ for $\alpha > 0$;
- $\Pi^\alpha_{\alpha+1} = \Sigma^\alpha_\alpha \cup \Pi^\alpha_\alpha \cup \{\langle \alpha \rangle_\alpha^\square \varphi(\dot{x}) \mid \varphi(x) \in \text{Form}\}$ for $\alpha > 0$;
- $\Sigma^\lambda \triangleright := \Pi^\lambda \triangleright := \bigcup_{\alpha < \lambda} \Sigma^\alpha_\alpha$ for $\lambda \in \text{Lim}$. 

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Theorem/conjecture Let $T$ be any c.e. theory, let $\xi < \Lambda$ for a natural ordinal notation system, and let $A \subseteq \mathbb{N}$. The following are equivalent

1. $A$ is c.e. in $\emptyset(\xi)$;
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So all the stuff about Turing progressions can be generalized in a straightforward fashion.

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No longer runs out of phase
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No longer runs out of phase

Theorem/conjecture: So all the stuff about Turing progressions can be generalized in a straight-forward fashion.
Definition

Let $\Gamma$ be a class of formulas. For ordinals $\alpha, \beta < \Lambda$ and $T$ a c.e. theory we define $\beta^{-\text{RFN}}_T(\Gamma)$ to be the schema $[\beta]_T^\square \varphi \rightarrow \varphi$ for $\varphi \in \Gamma$.

Instead of writing $0^{-\text{RFN}}_T(\Gamma)$ we shall just write $\text{RFN}_T^\Lambda(\Gamma)$.

We can now easily state and prove various equivalences between consistency statements and reflection principles.
Let $T$ be a c.e. theory containing $\text{ECA}_0$.

1. $\text{ECA}_0 \vdash \text{RFN}^\Lambda_T(\Pi_{\alpha+1}^\square) \equiv \langle \alpha \rangle^\square_T \top$;

2. For $\beta \leq \alpha$, we have $\text{ECA}_0 \vdash \beta - \text{RFN}^\Lambda_T(\Pi_{\alpha+1}^\square) \equiv \langle \alpha \rangle^\square_T \top$;

3. For $\beta > \alpha$ we have that $\text{ECA}_0 \vdash \beta - \text{RFN}^\Lambda_T(\Pi_{\alpha+1}^\square) \equiv \langle \beta \rangle^\square_T \top$;

4. For $\beta > \alpha$ we have that $\text{ECA}_0 \vdash \beta - \text{RFN}^\Lambda_T(\Pi_{\alpha+1}^\square) \equiv \langle \max\{\alpha, \beta\} \rangle^\square_T \top$. 

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