A construction of Lebesgue measure

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When one faces up to the problem of constructing Lebesgue measure in the euclidean space $\mathbb{R}^n$, it is usual to turn to general Caratheodory's procedure which provides a positive measure from a given outer measure ([1], [3]). This approach is useful in courses when some other measures are going to be studied, but if the goal of the course is just to study Lebesgue integration of functions of one or several real variables, it would be convenient to use a different method which is less abstract.

In this work we present an alternative approach based upon the idea of extending the volume of n-dimensional intervals to wider classes of subsets of $\mathbb{R}^n$. The first task is to show that if a set can be paved with a sequence of pairwise disjoint intervals then the sum of the volumes of these intervals does depends only on that set. This is done by expressing the volume of a kind of intervals with integers or rational endpoints as cardinals of lattice points.

As open sets can be paved by such sequences of intervals, the volume can be extended in a natural way to the class of open sets. Then the measure of a given set is computed by excess, by considering open sets which contain it, which leads to Lebesgue outer measure. This set function is not countably additive, but just subadditive.

Regarding the fact that Lebesgue outer measure is additive on the class of closed sets, we arrive to the definition of Lebesgue measurable set: a set A is said to be Lebesgue measurable when a closed subset F and an open supersets G can be found with $G \setminus F$ of arbitrarily small measure. With this definition the structure theorem for Lebesgue measurable sets is plain, as well as the countably additivity of Lebesgue measure and the fact that the class of Lebesgue measurable sets is a σ-algebra are not too difficult to obtain.

1 Intervals and volume

An interval I of $\mathbb{R}^n$ or n-dimensional interval is the cartesian product of n bounded intervals J_i of $\mathbb{R}$, that is, $I = J_1 \times \cdots \times J_n$. In this way, 1-dimensional intervals are just bounded intervals of the real line $\mathbb{R}$, 2-dimensional intervals are plane rectangles with sides parales to the coordinate axis, ...

If we denote by a_i the left-hand endpoint of the interval J_i and by b_i the right-hand one, we refer to $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ as the endpoints of I. It is clear by definition that $a_i \leq b_i$ for every i.

Conversely, given $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ in $\mathbb{R}^n$ satisfying $a_i \leq b_i$ for every i, there are several n-dimensional intervals having $\mathbf{a}$ and $\mathbf{b}$ as its endpoints: in order to define each J_i there are 4 possible choices of the inequalities $<$ or $\leq$, which led at least to two of such intervals; in fact, if $a_i < b_i$ for every i, there are 4^n such intervals! If we always choose the inequality $<$, that is $J_i = (a_i, b_i)$, we obtain that I is the open n-dimensional interval with endpoints $\mathbf{a}$ and $\mathbf{b}$. If we always choose the inequality $\leq$, that is $J_i = [a_i, b_i]$, then I will be the closed n-dimensional interval with endpoints $\mathbf{a}$ and $\mathbf{b}$. Any open interval is always an open set and any closed interval is a closed bounded set, thus a compact set because of the Heine-Borel theorem.

Let us notice that the empty set can be obtained as an open interval with endpoints $\mathbf{a}$ and $\mathbf{b}$ having equal at least one coordinate. This is a particular case of a degenerated interval: I is a degenerated n-dimensional interval if $a_i = b_i$ for some i. Thus degenerated intervals are contained in affine subspaces of a lower dimension and they will have n-dimensional volume zero.

Now we define the volume of an n-dimensional interval in a natural way[1], so that if $n = 1$ the volume will be the lenght, if $n = 2$ the volume will be the area = large $\times$ width, and if $n = 3$, the volume will be the usual volume = large $\times$ width $\times$ height.

Definition 1.1. Let $I = J_1 \times \cdots \times J_n$ be an n-dimensional interval with endpoints $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$. The volume of I is defined as the product of the lenghts of the intervals J_i, that is,

$$\mathrm{vol}(I) = (b_1 - a_1) \dots (b_n - a_n).$$

Let us observe that the volume of any interval I is nonnegative, being $a_i \leq b_i$ for every i, and that I is degenerated if and only if $\mathrm{vol}(I) = 0$.

We introduce the following notation[2]: for a set $A \subset \mathbb{R}^n$, the characteristic function of A is defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.

We focus now our attention on a particular kind of n-dimensional intervals whose volumes can be computed in a very nice way. We assume that the coordinates of the endpoints $\mathbf{a}$ and $\mathbf{b}$ of I are integer numbers, that is, $a_i, b_i \in \mathbb{Z}$ for every i. We also assume that I is not degenerated and that $I = [a_1, b_1) \times \cdots \times [a_n, b_n)$. We shall say in short that I is an integer n-dimensional interval. Let us notice that $b_i - a_i = \sum_{c_i \in \mathbb{Z}} \chi_{J_i}(c_i)$ for each i. Therefore we have

$$\mathrm{vol}(I) = \sum_{c_1 \in \mathbb{Z}} \chi_{J_1}(c_1) \cdots \sum_{c_n \in \mathbb{Z}} \chi_{J_n}(c_n) = \sum_{\mathbf{c} \in \mathbb{Z}^n} \chi_{J_1}(c_1) \dots \chi_{J_n}(c_n) = \sum_{\mathbf{c} \in \mathbb{Z}^n} \chi_I(\mathbf{c}),$$

[1] It could be convenient to comment to the students that the definition is necessary if it is assumed that the volume of the unit interval $[0,1)^n$ is 1 and that the volume is additive and invariant under translations.

[2] We can avoid the use of characteristic functions, using instead some facts on the cardinalities of finite sets: cardinality of a cartesian product, of finite unions, ...

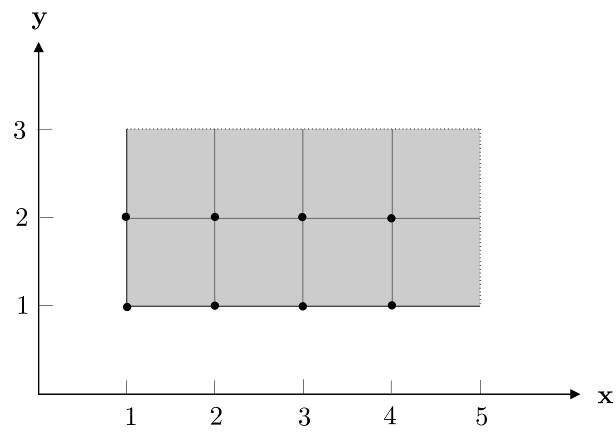

Figure 1: Volume of an integer 2-dimensional interval

that is, $\mathrm{vol}(I)$ equals the number of vectors with integers coordinates which lay inside I (see figure 1).

Now we consider non-degenerated intervals $I = [a_1, b_i) \times \cdots \times [a_n, b_n)$ whose endpoints $\mathbf{a}$ and $\mathbf{b}$ have rational coordinates. These intervals will be called rational n-dimensional intervals.

To compute the volume of rational intervals, let us see first how the volume changes under an homothetic transformation of ratio $\alpha > 0$. Let I be an n-dimensional interval with endpoints $\mathbf{a}$ and $\mathbf{b}$. Then the homothetic set $\alpha I = \{\alpha\mathbf{x} : \mathbf{x} \in I\}$ is also an interval; its endpoints are $\alpha\mathbf{a}$ and $\alpha\mathbf{b}$. Therefore we obtain

$$\mathrm{vol}(\alpha I) = \alpha^n \mathrm{vol}(I),$$

where the exponent n in the former formula is the dimension of the euclidean space we are[3].

It follows that, for a rational interval I and a positive integer q multiple of the denominator of every coordinate of the endpoints of I, so that qI is an integer interval,

$$\mathrm{vol}(I) = \frac{1}{q^n} \sum_{\mathbf{c} \in \mathbb{Z}^n} \chi_{qI}(\mathbf{c}).$$

We prove now a fundamental inequality on sums of volumes of intervals. We recall that a family of sets is said to be pairwise disjoint whenever any two different sets in the family are disjoint.

Lemma 1.2. *Let $\{I_1, \ldots, I_m\}$ and $\{H_1, \ldots, H_p\}$ be two finite families of rational n-dimensional intervals. Assume that the family $\{I_1, \ldots, I_m\}$ is pairwise*

[3] Some draws can help the student to understand why the dimension appears in this formula.

disjoint and that $I_1 \cup \cdots \cup I_m \subset H_1 \cup \cdots \cup H_p$. *Then,*

$$\sum_{k=1}^{m} \operatorname{vol}(I_k) \leq \sum_{i=1}^{p} \operatorname{vol}(H_i).$$

Proof. Let q be such that that qI_k and qH_i are integer intervals. Let $I^* = \bigcup_{k=1}^n I_k$ and $H^* = \bigcup_{i=1}^p H_i$. We have

$$\sum_{k=1}^{m} \operatorname{vol}(I_k) = \frac{1}{q^n} \sum_{k=1}^{m} \operatorname{vol}(qI_k) = \frac{1}{q^n} \sum_{k=1}^{m} \sum_{\mathbf{c} \in \mathbb{Z}^n} \chi_{qI_k}(\mathbf{c}) = \frac{1}{q^n} \sum_{\mathbf{c} \in \mathbb{Z}^n} \chi_{qI^*}(\mathbf{c})$$

$$\leq \frac{1}{q^n} \sum_{\mathbf{c} \in \mathbb{Z}^n} \chi_{qH^*}(\mathbf{c}) \leq \frac{1}{q^n} \sum_{i=1}^{p} \sum_{\mathbf{c} \in \mathbb{Z}^n} \chi_{qH_i}(\mathbf{c}) = \sum_{i=1}^{p} \operatorname{vol}(H_i).$$

□

To extend the property stablished in Lemma 1.2 to arbitrary intervals we need the next approximation result:

Lemma 1.3. *Let I be an n-dimensional interval. Let $\varepsilon > 0$. Then there exist a rational interval H whose interior contains I and* $\operatorname{vol}(H) - \operatorname{vol}(I) < \varepsilon$. *If moreover I is non degenerated then there exists also another rational interval J such that the closure of J is contained in I and* $\operatorname{vol}(I) - \operatorname{vol}(J) < \varepsilon$.

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ the endpoints of the interval I. For $\delta > 0$ we define the interval $L = \prod_{i=1}^n (a_i - \delta, b_i + \delta)$. It is clear that $\operatorname{vol}(L) = \prod_{i=1}^n (b_i - a_i - 2\delta)$ is a continuous function of δ which matches up with $\operatorname{vol}(I)$ for $\delta = 0$. So, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\operatorname{vol}(I) - \operatorname{vol}(L) < \varepsilon$. We choose rational numbers $x_i \in (a_i - \delta, a_i)$ and $y_i \in (b_i, b_i + \delta)$ and we define the rational interval $J = [x_1, y_1) \times \cdots \times [x_n, y_n)$. It is obvious that $[a_i, b_i] \subset (x_i, y_i)$ for every i, thus H contains the closure of I and $\operatorname{vol}(H) - \operatorname{vol}(I) \leq \operatorname{vol}(L) - \operatorname{vol}(I) < \varepsilon$.

If I is non degenerated the interval H is constructed in a similar way by considering this time $L = \prod_{i=1}^n (a_i + \delta, b_i - \delta)$ for $\delta > 0$ small enough, $0 < \delta < \min_{1 \leq i \leq n} (b_i - a_i)/2$. □

The following proposition is the key to define the measure of the open sets and to prove some of the main properties of Lebesgue measure[4]. In its proof the student will find, perhaps for the first time, the necessity of adding infinite errors. This is done with the use of the geometrical series $\varepsilon = \sum_{k=1}^{\infty} \varepsilon/2^k$.

Proposition 1.4. *Let (I_k) and (H_i) two countable families of intervals such that $\bigcup_{k=1}^{\infty} I_k \subset \bigcup_{i=1}^{\infty} H_i$. If the family (I_k) is pairwise disjoint then*

$$\sum_{k=1}^{\infty} \operatorname{vol}(I_k) \leq \sum_{i=1}^{\infty} \operatorname{vol}(H_i).$$

[4] It is worth to notice that some similar result has to be proved also in other constructions when one wants to show that Lebesgue measure agrees with the volume on intervals

Proof. We can assume that all the intervals I_k are non degenerated because the volume of such an interval is zero. Let $\varepsilon > 0$. Lemma 1.3 allows us to choose, for each k and i, rational intervals J_k and L_i, such that the closure of J_k is contained in I_k and H_i is contained in the interior of L_i, satisfying $\mathrm{vol}(I_k) - \mathrm{vol}(J_k) \le \varepsilon/2^k$ and $\mathrm{vol}(L_i) - \mathrm{vol}(H_i) \le \varepsilon/2^i$.

Fix $m \ge 1$. The family (L_i) is an open covering of the compact set $\bigcup_{k=1}^m J_k$, hence there exists $p \ge 1$ such that $\bigcup_{k=1}^m J_k \subset \bigcup_{i=1}^p L_i$. By Lemma 1.2 we have

$$\begin{aligned}\sum_{k=1}^m \mathrm{vol}(I_k) &\le \sum_{k=1}^m \Big(\mathrm{vol}(J_k) + \frac{\varepsilon}{2^k}\Big) \le \sum_{k=1}^m \mathrm{vol}(J_k) + \varepsilon \\ &\le \sum_{i=1}^p \mathrm{vol}(L_i) + \varepsilon \le \sum_{i=1}^p \Big(\mathrm{vol}(H_i) + \frac{\varepsilon}{2^k}\Big) + \varepsilon \le \sum_{i=1}^\infty \mathrm{vol}(H_i) + 2\varepsilon.\end{aligned}$$

Letting $m \to \infty$ and $\varepsilon \to 0$ we get the desired inequality. □

Remark 1.5. Proposition 1.4 is customary proved by using refined partitions of the given intervals ([2], [4]). Our aproach is inspired in the fact that characteristic functions of intervals are Riemann integrable functions, thus their integrals which *a fortiori* are their volumes, can be computed as

$$\mathrm{vol}(I) = \lim_{\varepsilon\to 0} \varepsilon^n \sum_{\mathbf{c}\in\mathbb{Z}^n} \chi_I(\varepsilon\mathbf{c}) = \lim_{\varepsilon\to 0} \varepsilon^n \#(I \cap \varepsilon\mathbb{Z}^n).$$

This equality was obtained by L. Rodríguez-Piazza without appealing to Riemann integration, and applied to obtain Lemma 1.2.

2 Measure of open sets and Lebesgue outer measure

Any open set G can be paved with intervals. That is to say, there is a pairwise disjoint sequence of intervals (I_k) such that $G = \bigcup_{k=1}^\infty I_k$. To prove this for non-empty open sets we need some preliminaries on dyadic cubes. Given $k \ge 0$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, the interval

$$I_k^{\mathbf{a}} = \Big[\frac{a_1}{2^k}, \frac{a_1+1}{2^k}\Big) \times \cdots \times \Big[\frac{a_n}{2^k}, \frac{a_n+1}{2^k}\Big)$$

is called a dyadic cube. Thus dyadic intervals are rational intervals. It is clear that $\mathbb{R}^n = \bigcup_{\mathbf{a}\in\mathbb{Z}^n} I_k^{\mathbf{a}}$ for each k.

Let $\mathbf{x} \in I_k^{\mathbf{a}} \cap I_j^{\mathbf{b}}$ and let $j \le k$. Then $2^{-k}a_i \le x_i < 2^{-j}(b_i+1)$ for each i, hence $a_i < 2^{k-j}(b_i+1)$, and as they are integer numbers, $a_i + 1 \le 2^{k-j}(b_i+1)$, hence $2^{-k}(a_i+1) \le 2^{-j}(b_i+1)$. In a similar way, starting from $2^{-j}b_i \le x_i < 2^{-k}(a_i+1)$, we see that $2^{-j}b_i \le 2^{-k}a_i$. Therefore we obtain that $I_k^{\mathbf{a}} \subset I_j^{\mathbf{b}}$.

In particular, if $\mathbf{a} \ne \mathbf{b}$ then $I_k^{\mathbf{a}} \cap I_k^{\mathbf{b}} = \emptyset$, hence $(I_k^{\mathbf{a}})_{\mathbf{a}\in\mathbb{Z}^n}$ is a countable partition of $\mathbb{R}^n$ for each k. And if $j < k$ then $I_k^{\mathbf{a}}$ and $I_j^{\mathbf{b}}$ are either disjoint or $I_k^{\mathbf{a}} \subset I_j^{\mathbf{b}}$.

Proposition 2.1. *Let $G \subset \mathbb{R}^n$ a nonempty open set. Then there exists a sequence (I_k) of pairwise disjoint dyadic cubes such that $G = \bigcup_{k=1}^{\infty} I_k$.*

Proof. We consider $D_0 = \{\mathbf{a} \in \mathbb{Z}^n \colon \ I_0^{\mathbf{a}} \subset G\}$, selecting the family of dyadic cubes $I_0^{\mathbf{a}}$ whose closures $I_0^{\mathbf{a}}$ are contained in G. Then we consider $D_1 = \{\mathbf{a} \in \mathbb{Z}^n \colon I_1^{\mathbf{a}} \subset G, \ I_0^{\mathbf{b}} \cap I_1^{\mathbf{a}} = \emptyset \ \text{ for every } \ \mathbf{b} \in D_0\}$, selecting this time those dyadic cubes $I_1^{\mathbf{a}}$ with closure contained in G which are not contained in any of the selected in the first step. In general, let $D_k = \{\mathbf{a} \in \mathbb{Z}^n : \ I_k^{\mathbf{a}} \subset G, \ I_j^{\mathbf{b}} \cap I_k^{\mathbf{a}} = \emptyset, \text{ for all } \mathbf{b} \in D_j \text{ and all } j < k\}$.

Then $G = \bigcup_{k=0}^{\infty} \bigcup_{\mathbf{a} \in D_k} I_k^{\mathbf{a}}$. Indeed, if $\mathbf{x} \in G$, we take k such that the diameter of any cube $I_k^{\mathbf{a}}$ is strictly less than $d(\mathbf{x}, G^c)$. Next, we choose $\mathbf{a} \in \mathbb{Z}^n$ such that $\mathbf{x} \in I_{\mathbf{a}}^k$. Then $I_k^{\mathbf{a}} \subset G$ and, either $\mathbf{a} \in D_k$, in which case $\mathbf{x}$ belongs to the union, or $\mathbf{a} \notin D_k$; in this case there must exist $j < k$ and $\mathbf{b} \in D_j$ with $I_{\mathbf{b}}^j \cap I_{\mathbf{a}}^k \neq \emptyset$, so $I_k^{\mathbf{a}} \subset I_j^{\mathbf{b}}$ and then $\mathbf{x} \in I_j^{\mathbf{b}}$, belonging $\mathbf{x}$ also in this case to the union. ☐

Propositions 1.4 and 2.1 allow us to measure open sets[5]:

Definition 2.2. Let G be an open subset of $\mathbb{R}^n$. Let (I_k) be a pairwise sequence of intervals covering G. The measure of G is defined as

$$\mathfrak{m}(G) = \sum_{k=1}^{\infty} \operatorname{vol}(I_k).$$

Let us observe that $\mathfrak{m}(\emptyset) = 0$ and that $\mathfrak{m}(I) = \operatorname{vol}(I)$ if I is an open interval.

Lemma 2.3. *The function $\mathfrak{m}$ defined on the open sets satisfies:*

1. *It is non decreasing, that is, if $G_1 \subset G_2$ are open sets, then $\mathfrak{m}(G_1) \leq \mathfrak{m}(G_2)$.*
2. *It is countably subadditive, that is, if (G_k) is a sequence of open sets then $\mathfrak{m}(\bigcup_{k=1}^{\infty} G_k) \leq \sum_{k=1}^{\infty} \mathfrak{m}(G_k)$.*
3. *It is countably additive, that is, if (G_k) is a pairwise disjoint sequence of open sets, then $\mathfrak{m}(\bigcup_{k=1}^{\infty} G_k) = \sum_{k=1}^{\infty} \mathfrak{m}(G_k)$.*

Proof. Proposition 1.4 gives directly the first property. For the other two properties, take $(I_{k,j})$ according to Definition 2.2 such that $G_k = \bigcup_{i=1}^{\infty} I_{k,i}$ and $\mathfrak{m}(G_k) = \sum_{i=1}^{\infty} \operatorname{vol}(I_{k,i})$, for every k. If (I_j) is a pairwise disjoint covering of $\bigcup_{k=1}^{\infty} G_k$ then $\bigcup_{j=1}^{\infty} I_j \subset \bigcup_{i,j=1}^{\infty} I_{k,j}$ thus Proposition 1.4 implies

$$\sum_{j=1}^{\infty} \operatorname{vol}(I_j) \leq \sum_{i,j=1}^{\infty} \operatorname{vol}(I_{k,j}) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \operatorname{vol}(I_{k,i}) \right),$$

hence we have the second property. For the third one, let us observe that if the sequence (G_k) is pairwise disjoint then the inequality in the former equation becomes an equality. ☐

[5] Again it coul be convenient to notice that the given definition is necessary if one ask the measure to be countably additive and to be an extension of the volume.

We are ready to define Lebesgue outer measure:

Definition 2.4. Let $A \subset \mathbb{R}^n$. We define the Lebesgue outer measure of A as

$$\mathfrak{m}^*(A) = \inf\{\mathfrak{m}(G)\colon\ A \subset G \subset \mathbb{R}^n,\ G \text{ open}\}.$$

Let us recall that an outer measure is a function defined for every subset, which is non decreasing, subbaditive and asigns the value zero to the empty set.

Proposition 2.5. *The function $\mathfrak{m}^*$ is an outer measure on $\mathbb{R}^n$. For intervals I we have $\mathfrak{m}^*(I) = \mathrm{vol}(I)$ and for open sets G we have $\mathfrak{m}^*(G) = \mathfrak{m}(G)$.*

Proof. It is clear that $\mathfrak{m}^*(\emptyset) = 0$ and that $A \subset B \subset \mathbb{R}^n$ implies $\mathfrak{m}^*(A) \leq \mathfrak{m}^*(B)$. To prove subadditivity, let (A_k) be a sequence of subsets of $\mathbb{R}^n$ and let $A = \bigcup_{k=1}^{\infty} A_k$. Given $\varepsilon > 0$, there exists an open set $G_k \supset A_k$ such that $\mathfrak{m}^*(G_k) \leq \mathfrak{m}^*(A_k) + \varepsilon/2^k$. By Lemma 2.3 we have

$$\mathfrak{m}^*(A) \leq \mathfrak{m}\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \mathfrak{m}(G_k) \leq \sum_{k=1}^{\infty}\left(\mathfrak{m}^*(A_k) + \frac{\varepsilon}{2^k}\right) = \sum_{k=1}^{\infty} \mathfrak{m}^*(A_k) + \varepsilon.$$

It is enough to make $\varepsilon \to 0$ to obtain $\mathfrak{m}^*(A) \leq \sum_{k=1}^{\infty} \mathfrak{m}^*(A_k)$.

If G is an open set then $\mathfrak{m}(B) \geq \mathfrak{m}(G)$, for every open set $B \supset G$ since Lemma 2.3, hence $\mathfrak{m}^*(G) \geq \mathfrak{m}(G)$. The reverse inequality is plain by definition.

In particular, if I is an open interval $\mathfrak{m}^*(I) = \mathfrak{m}(I) = \mathrm{vol}(I)$. For an arbitrary interval J, according to Lemma 1.3, there exists a sequence of open intervals $I_k \supset J$ with $\mathrm{vol}(I_k) \to \mathrm{vol}(J)$. As $\mathfrak{m}^*(J) \leq \mathrm{vol}(I_k)$ for every k we obtain $\mathfrak{m}^*(J) \leq \mathrm{vol}(J)$. For the reverse inequality, observe that if I is the open interval which is the interior of J, then $\mathrm{vol}(J) = \mathrm{vol}(I) = \mathfrak{m}^*(I) \leq \mathfrak{m}^*(J)$. □

It follows from this proposition and Lemma 2.3 that $\mathfrak{m}^*$ is countably additive on open sets. We shall show in the next Section that $\mathfrak{m}^*$ is countably additive on a very much larger family of sets, the Lebesgue measurable sets. We prove now that it is finitely additive on closed sets. We shall denote by $d(\mathbf{x}, A) = \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in A\}$ the distance between a point $\mathbf{x}$ and a set A. As $|d(\mathbf{x}, A) - d(\mathbf{y}, A)| \leq d(\mathbf{x}, \mathbf{y})$, the function $d(\mathbf{x}, A)$ is a continuous function of $\mathbf{x}$.

Lemma 2.6. *Let $\{A_1, \ldots, A_p\}$ be a pairwise disjoint family of closed subsets of $\mathbb{R}^n$. Then*

$$\mathfrak{m}^*\left(\bigcup_{k=1}^{p} A_k\right) = \sum_{k=1}^{p} \mathfrak{m}^*(A_k).$$

Proof. Let $G_k = \{\mathbf{x} \in \mathbb{R}^n\colon\ d(\mathbf{x}, A_k) < d(\mathbf{x}, \bigcup_{j \neq k} A_j)\}$. Then $\{G_1, \ldots, G_p\}$ is a pairwise disjoint family of open sets such that $A_k \subset G_k$ for every k.

Let G be any open set such that $\bigcup_{k=1}^{p} A_k \subset G$. Then,

$$\sum_{k=1}^{p} \mathfrak{m}^*(A_k) \leq \sum_{k=1}^{p} \mathfrak{m}(G_k \cap G) = \mathfrak{m}\left(\bigcup_{k=1}^{p} G_k \cap G\right) \leq \mathfrak{m}(G),$$

hence we derive that $\sum_{k=1}^{p} \mathfrak{m}^*(A_k) \leq \mathfrak{m}^*(\bigcup_{k=1}^{p} A_k)$. As the reverse inequality follows from the subadditivity of $\mathfrak{m}^*$, the proof is complete. □

3 Lebesgue measurable sets and Lebesgue measure

Assume that I is an n-dimensional non degenerated interval. Given $\varepsilon > 0$, by Lemma 1.3 there exists a closed interval J and an open interval H with $J \subset I \subset H$ and $\mathrm{vol}(H) - \mathrm{vol}(J) < \varepsilon$. Then $\mathrm{vol}(H) = \mathrm{vol}(J) + \mathfrak{m}(H \setminus J)$ because if (I_k) is a pairwise disjoint sequence of intervals covering the open set $H \setminus J$, then $\mathrm{vol}(H)$ can be computed using the covering obtained by adding the interval J to the sequence (I_k). Hence $\mathfrak{m}(H \setminus J) < \varepsilon$. This property leads us to introduce the following definition:

Definition 3.1. A set $A \subset \mathbb{R}^n$ is said to be Lebesgue measurable if for each $\varepsilon > 0$ there exist a closed set F and an open set G such that $F \subset A \subset G$ and $\mathfrak{m}(G \setminus F) < \varepsilon$.

Thus every non degenerated interval is measurable. It is also clear by definition that every set of measure zero is measurable. In particular, every degenerated interval I is measurable because $\mathfrak{m}^*(I) = 0$.

The restriction of $\mathfrak{m}^*$ to the class of Lebesgue measurable sets is called the Lebesgue measure, denoted by $\mathfrak{m}$. We shall see that $\mathfrak{m}$ is a positive measure extending the volume on the intervals and the measure on the open sets.

Lemma 3.2. *Let (A_k) be a sequence of measurable sets.*

1. *If $\sum_{k=1}^{\infty} \mathfrak{m}^*(A_k) < \infty$ then $\bigcup_{k=1}^{\infty} A_k$ is measurable.*
2. *The set $\bigcup_{k=1}^{p} A_k$ is measurable for every p.*
3. *If the sequence is pairwise disjoint then $\mathfrak{m}^* \left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1} \mathfrak{m}^*(A_k)$.*

Proof. Let $\varepsilon > 0$. For each k, we take closed sets F_k and open sets G_k with $F_k \subset A_k \subset G_k$ and such that $\mathfrak{m}(G_k \setminus F_k) < \varepsilon/2^k$. Let $A = \bigcup_{k=1}^{\infty} A_k$.

If $\sum_{k=1}^{\infty} \mathfrak{m}^*(A_k) < \infty$ we choose p such that $\sum_{k=p+1}^{\infty} \mathfrak{m}^*(A_k) < \varepsilon$, hence $\sum_{k=p+1}^{\infty} \mathfrak{m}^*(G_k) < 2\varepsilon$ being $\mathfrak{m}^*(G_k) < \mathfrak{m}^*(A_k) + \mathfrak{m}^*(G_k \setminus A_k)$. Then $F = \bigcup_{k=1}^{p} F_k$ is a closed subset of A, $G = \bigcup_{k=1}^{\infty} G_k$ is an open set containing A and

$$\mathfrak{m}^*(G \setminus F) \leq \mathfrak{m}^* \left(\bigcup_{k=1}^{p} G_k \setminus \bigcup_{k=1}^{p} F_k \right) + \mathfrak{m}^* \left(\bigcup_{k=p+1}^{\infty} G_k \right)$$
$$\leq \sum_{k=1}^{p} \mathfrak{m}^*(G_k \setminus F_k) + \sum_{k=p+1}^{\infty} \mathfrak{m}^*(G_k) < 3\varepsilon,$$

which shows that A is measurable.

The second statement follows by letting $A_k = \emptyset$ for $k > p$, so that we have $\sum_{k=p+1}^{\infty} \mathfrak{m}^*(A_k) < \varepsilon$ for every $\varepsilon > 0$, which was exactly what we needed to show that A was measurable.

To show the third statement, observe that $\mathfrak{m}^*(A_k) \leq \mathfrak{m}^*(F_k) + \varepsilon/2^k$. It follows from Lemma 2.6 that for every p

$$\sum_{k=1}^{p} \mathfrak{m}^*(A_k) \leq \sum_{k=1}^{p} \left(\mathfrak{m}^*(F_k) + \frac{\varepsilon}{2^k} \right) = \mathfrak{m}^* \left(\bigcup_{k=1}^{p} F_k \right) + \sum_{k=1}^{p} \frac{\varepsilon}{2^k} \leq \mathfrak{m}^* \left(A \right) + \varepsilon,$$

Letting $\varepsilon \to 0$ and $p \to \infty$ we obtain $\sum_{k=1}^{\infty} \mathfrak{m}^*(A_k) \leq \mathfrak{m}^* \left(A \right)$, which finishes the proof regarding that $\mathfrak{m}^*$ is subadditive. □

We recall that a σ-algebra on $\mathbb{R}^n$ is a family of subsets of $\mathbb{R}^n$ which contains the empty set and is closed under complementation and countable union. Let us notice that we know already that the empty set is measurable and the second part of Lemma 3.2 gives that the class of measurable sets is closed under finite unions.

Theorem 3.3. *The class of Lebesgue measurable sets is a σ-algebra on $\mathbb{R}^n$ and the restriction $\mathfrak{m}$ of Lebesgue outer measure $\mathfrak{m}^*$ to this class is a positive measure.*

Proof. First we show that if A is measurable then the complement A^c so is: given $\varepsilon > 0$ we take a closed set F and an open set G satisfying $\mathfrak{m}(G \setminus F) < \varepsilon$ and $F \subset A \subset G$. Then $G^c \subset A \subset F^c$, G^c is closed, F^c is open and $\mathfrak{m}(F^c \setminus G^c) = \mathfrak{m}(G \setminus F) < \varepsilon$.

Let (A_k) be a sequence of measurable sets. Then $A = \bigcup_{k=1}^{\infty} A_k$ can be written as $A = A_1 \cup (A_1 \cup A_2) \cup \ldots$, an increasing union of measurable sets. Thus, to prove that A is measurable we could assume that (A_k) is an increasing sequence of measurable sets. Moreover, as $A = A_1 \cup (A_2 \setminus A_1) \cup \ldots$ and the set $A_{k+1} \setminus A_k = A_{k+1} \cap A_k^c = (A_{k+1}^c \cup A_k)^c$ is measurable, we shall assume that (A_k) is a pairwise disjoint sequence of measurable sets.

With this assumption, if $\mathfrak{m}^*(A) < \infty$ it follows directly from Lemma 3.2 that A is measurable. In general, let us denote by I_p the n-dimensional interval $(-p,p)^n$. Then $B_p = A \cap (I_{p+1} \setminus I_p) = \bigcup_{k=1}^{\infty} A_k \cap (I_{p+1} \setminus I_p)$ is measurable since $\mathfrak{m}^*(B_p) \leq \mathrm{vol}(I_{p+1}) < +\infty$. Thus we can choose, for each p, a closed set $F_p \subset B_p$ and an open set $B_p \subset G_p$ with $\mathfrak{m}(G_p \setminus F_p) < \varepsilon/2^p$. Let $F = \bigcup_{p=1}^{\infty} F_p$ and $G = \bigcup_{p=1}^{\infty} G_p$. Then F is closed because any convergent sequence in F must be contained in finitely many F_p, otherwise it could not be bounded! Therefore

$$\mathfrak{m}(G \setminus F) \leq \mathfrak{m} \left(\bigcup_{p=1}^{\infty} G_p \setminus F_p \right) \leq \sum_{p=1}^{\infty} \mathfrak{m}(G_p \setminus F_p) < \varepsilon$$

and we obtain that A is measurable.

Finally, that $\mathfrak{m}$ is countably additive on the class of measurable sets is the second statement in Lemma 3.2. □

As any interval is measurable it follows that open sets are measurable. Thus every closed set is measurable too. Since we do not leave the class of measurable sets by taking countable unions, we have that $\mathcal{F}_\sigma$-sets (countable unions of closed sets) are Lebesgue measurable. The complement of such a set is called a $\mathcal{G}_\delta$-set (countable intersection of open sets) and is measurable. We finish by showing the following structure theorem for Lebesgue measurable sets:

Theorem 3.4. *For a subset A of $\mathbb{R}^n$ the following conditions are equivalent:*

1. *The set A is Lebesgue measurable.*
2. *There exists a $\mathcal{F}_\sigma$-set $B \subset A$ with $\mathfrak{m}^*(A \setminus B) = 0$.*
3. *There exists a $\mathcal{G}_\delta$-set $C \supset A$ with $\mathfrak{m}^*(C \setminus A) = 0$.*

Proof. We see that the second and the third conditions are equivalent by taking complementary. As $\mathcal{F}_\sigma$-sets and sets of outer measure zero are Lebesgue measurable we obtain that the first condition is a consequence of the second one. Finally, if a set A is measurable then there exists a sequence of closed sets $F_k \subset A$ with $\mathfrak{m}^*(A \setminus F_k) \to 0$. Hence $B = \bigcup_{k=1}^{\infty} F_k \subset A$ is an $\mathcal{F}_\sigma$-set with $\mathfrak{m}^*(A \setminus B) \le \mathfrak{m}^*(A \setminus F_k)$ for every k, and it follows that $\mathfrak{m}^*(A \setminus B) = 0$. □

Remark 3.5. Another possible choice for the definition of measurability is the following: A is Lebesgue measurable if for each $\varepsilon > 0$ there exist a closed set $F \subset A$ and $\mathfrak{m}^*(A \setminus F) < \varepsilon$. With this definition Lemma 3.2 can be proved in the same way. But to show that the complement of such a set is also measurable we need to show first that every open bounded set is measurable (use Lemma 3.2, Proposition 2.1 and the fact that every interval is measurable) and derive that closed sets F has the following property: for each $\varepsilon > 0$ there exists an open set $F \subset G$ such that $\mathfrak{m}(G \setminus F) < \varepsilon$. This is more involved than the way we choosed.

References

[1] Cohn, D. L. *Measure theory.* Birkhauser, Boston, 1980.

[2] Jones, F. *Lebesgue Integration on Euclidean Space.* Jones and Bartlett Publ. Int., Londres, 1993.

[3] Rudin, W. *Real and Complex Analysis.* Tata McGraw-Hill, New Delhi, 1985.

[4] Sikorski, R. *Advanced Calculus: Functions of several variables.* Polish Scintific Publishers, Varsovia, 1969.