Becoming Aware of Propositional Variables

Hans van Ditmarsch[1,⋆] and Tim French[2]

[1] University of Sevilla, Sevilla, Spain
hvd@us.es
[2] University of Western Australia, Perth, Australia
tim@csse.uwa.edu.au

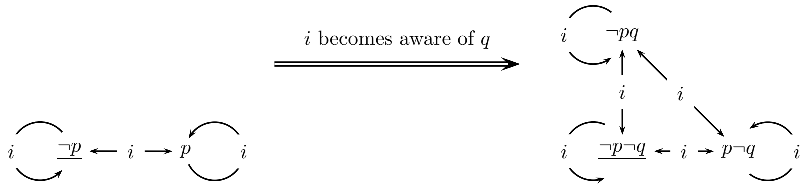
Abstract. We examine a logic that combines knowledge, awareness, and change of awareness. Change of awareness involves that an agent becomes aware of propositional variables. We show that the logic is decidable, and we present a complete axiomatization.

1 Introduction

Awareness and knowledge. Modal logic has long been used to reason about knowledge and belief in multi-agent systems. In modal logics we model *uncertainty* by allowing the value of propositions to vary between the so-called possible worlds. An agent *knows* a proposition in a given world if the proposition is true in all worlds accessible from that world. The logics require that the agents are *aware* of all propositional variables in the model. Thus reasoning in these models is undertaken under a closed world assumption: the relevant propositional variables are known to all agents. For every propositional variable in every world, every agent assigns a value to that variable.

While agents may be *uncertain* about the value of propositions, they may also be *unaware* of these propositions, and they may *become aware* of propositions. Uncertainty and incompleteness (i.e., unawareness) are different issues in modelling multi-agent systems. Without taking awareness into account, it seems difficult to explain the following transition, wherein the epistemic complexity of the model increases:

> *Initially, Hans (i) does not know whether coffee is served (p) after his talk. (Actually, no coffee will be served—$\neg p$, underlined.) Hans is unaware of it that wine is not served ($\neg q$) after his talk. Now, someone mentions that wine and coffee will not both be served. This makes Hans aware that wine is an issue. After this, Hans does not know whether coffee is served after his talk and also does not know whether wine is served after his talk. (Of course, actually, there is no coffee and no wine.)*



[⋆] Hans van Ditmarsch is also affiliated to the Institute of Mathematical Sciences Chennai as a research associate.

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We find that there are many subtleties and intricacies involved in defining the semantics for such dynamics of awareness. In this paper we will discuss these intricacies and in doing so make the following contributions:

1. We will introduce a new form of model equivalence modulo the agents' awareness and uncertainty, called *awareness bisimulation.*
2. We will define a new type of knowledge, referred to as *intrinsic knowledge.* Intrinsic knowledge is essential to express the dynamic interactions between awareness and knowledge. It relates to implicit and explicit knowledge.
3. We will introduce an logical operator for *becoming aware* of propositional variables and give semantics for this operator that is consistent with our intuitions of awareness and knowledge.

Prior research. Our work is rooted in: the tradition of epistemic logic [10] and in particular multi-agent epistemic logic [13,2]; in various research since the 1980s on the interaction of awareness and knowledge [1,14,15,8] — including a relation to recent works like [9,5,7]; and in modal logical research in propositional quantification, starting in the 1970s with [3] and followed up by work on bisimulation quantifiers [18,11,4].

Works treating awareness either follow a more *semantically* flavoured approach, where awareness concerns propositional variables in the valuation [1,15,8], or a more *syntactically* flavoured approach. In the latter, awareness concerns all formulas of the language in a given set, in order to model 'limited rationality' of agents. It is (also) pursued in [1] and in recent work like [5]. We are straight into the semantic corner: within the limits of their awareness, agents are fully rational.

For the static part of the logical language we follow [1]. For the dynamic part, it is remarkable that levels of 'interactive unawareness' in [8] can be described in terms of the awareness bisimulation introduced in our work (at the end of our paper). The insights made clear in their paper were very motivating for us. Our work builds on [17], which focusses on a special case (public global awareness) of the current paper, but unlike the present paper also treats awareness of other agents and forgetting (i.e., becoming unaware).

2 Structures

Given are a countably infinite set of propositional variables (facts) P and a (disjoint) finite set of agents N. Propositional variables are named p, q, r, and agent variables are named i, j, k, possibly indexed or quoted.

Definition 1 (Epistemic awareness model). *An* epistemic awareness model *for* N *and* P *is a tuple* $M = (S, R, \mathcal{A}, V)$ *that consists of a* domain S *of (factual)* states *(or 'worlds'), an* accessibility function $R : N \to \mathcal{P}(S \times S)$*, an* awareness function $\mathcal{A} : N \to S \to \mathcal{P}(P \cup N)$ *and a* valuation function $V : P \to \mathcal{P}(S)$*. For* $R(i)$ *we write* R_i *and for* $\mathcal{A}(i)$ *we write* $\mathcal{A}_i$*; accessibility function* R *can be seen as a set of* accessibility relations R_i*, and* V *as a set of* valuations $V(p)$*. A pointed epistemic awareness model* (M, s) *is an* epistemic awareness state.

Given an arbitrary model M we will refer to the elements of the tuple as $(S^M, R^M, \mathcal{A}^M, V^M)$. The awareness function $\mathcal{A}$ may be varied to reflect different logics. *Public global awareness* results if the value of $\mathcal{A}$ is the same for all agents and for all states. *Individual global awareness* results if the awareness function is the same in all states, but may vary among agents. These logics are discussed in [17]. In this work we focus on the logic of *individual local awareness* where there are no constraints placed on the awareness function $\mathcal{A}$. For the sake of generality we will assume no restrictions on the accessibility function R_i, either. However, we will sometimes require that the relation satisfies some simple properties (such as reflexivity, transitivity, etc.). The property of *awareness introspection* [8] holds if all agents know when they are aware of a fact or of another agent: "If $(s,t),(s,u) \in R_i$, then $\mathcal{A}_i(t) = \mathcal{A}_i(u)$."

Awareness bisimulation. Consider the following scenario: in state s agent i is aware of proposition p, state u is accessible for agent i from state s, and in state u agent j is aware of proposition p and also of proposition q. That agent j is also aware of q in u should leave agent i indifferent, as she is not aware of q in s! This sort of similarity is captured in the following notion, named *awareness bisimulation.* Informally, given a model and a set of propositional variables $P' \subseteq P$, another model is a P' awareness bisimulation if it cannot be distinguished from the first by formulas consisting only of the propositional variables in P', in the scope of agents who are aware of those propositions.

Definition 2 (Awareness bisimulation). *Let epistemic awareness models* $M = (S, R, \mathcal{A}, V)$ *and* $M' = (S', R', \mathcal{A}', V')$ *be given. For all* $P' \subseteq P$ *we define the relation* $\mathfrak{R}[P']$ *by* $(s, s') \in \mathfrak{R}[P']$ *iff:*

atoms *for all* $p \in P'$*,* $s \in V(p)$ *iff* $s' \in V'(p)$*;*
aware *for all* $i \in N$*,* $\mathcal{A}_i(s) \cap P' = \mathcal{A}'_i(s') \cap P'$*;*
forth *for all* $i \in N$*, if* $t \in S$ *and* $R_i(s,t)$ *then there is a* $t' \in S'$ *such that* $R'_i(s',t')$ *and* $(t,t') \in \mathfrak{R}[P' \cap \mathcal{A}_i(s)]$*;*
back *for all* $i \in N$*, if* $t' \in S'$ *and* $R'_i(s',t')$ *then there is a* $t \in S$ *such that* $R_i(s,t)$ *and* $(t,t') \in \mathfrak{R}[P' \cap \mathcal{A}'_i(s')]$*.*

Epistemic awareness state (M', s') *is a* P'-awareness bisimulation *of epistemic awareness state* (M, s) *(written* $(M', s') \leftrightarrows_{P'} (M, s)$*) iff* $(s, s') \in \mathfrak{R}[P']$*.*

The 'aware' clause can be considered as an additional basic structural requirement besides 'atoms', only due to the nature of our models where states have more structure than merely factual truth. If we were to replace $\mathfrak{R}[P' \cap \mathcal{A}_i(s)]$ in the **back** and **forth** clauses with $\mathfrak{R}[P']$, we would have the definition of a standard (restricted) bisimulation over labelled transition structures [16]. (Restricted to $P' \subseteq P$.) Thus every bisimulation is an awareness bisimulation. Vice versa, if all agents are aware of all propositional variables, the awareness bisimulation is a standard bisimulation (for the relations R_i). This is what we desire: we then revert to the standard multi-agent epistemic situation, where awareness plays no role.

Proposition 1. *The relation $\leftrightarrows_{P'}$ is an equivalence relation.*

Proof. This can be easily seen by examining the Definition 2.

Definition 2 is more complex than the definition of standard bisimulation, however its motivation is very simple. Two worlds are P'-awareness bisimilar if, for any observer aware only of the propositions in P', the worlds appear identical. It gives us the "P'-perspective" of a world. We also call it *observational equivalence.* Let that observer be agent i in state s, then the required P' is $\mathcal{A}_i(s)$ and her perspective is that of $\mathcal{A}_i(s)$-awareness-bisimilarity. We might also say that her view of the model is that of its $\mathfrak{R}[\mathcal{A}_i(s)]$ equivalence class.

The crucial part of the definition is that in 'forth', in the requirement "$(t, t') \in \mathfrak{R}[P' \cap \mathcal{A}_i(s)]$", the bisimulation for **state t** is (further) restricted to the propositional variables that agent i is aware of in **state s**, the i-predecessor of t. (And similarly for 'back'.) An honoured principle (also in economics, and in artificial intelligence) is that incompleteness precedes uncertainty. The awareness function of an agent in a given state (incompleteness) determines what the agent can 'see' in all accessible states (uncertainty), *and so on.* This chaining of awareness is expressed with awareness bisimulation. This chaining requirement was present in epistemic awareness structures since its inception in [1]. We have merely employed it to the full and in the *one and only* way, for structural similarity.

Example. In Figure 1 agent i is aware of p but unaware of q in state s. In the figure, names of states are followed, separated by a dot, by values of propositional variables. Unaware variables are between parentheses. For example, $s.p(\neg q)$ means that in state s p is true and q is false, and the agent is aware of p and not of q. The three depicted epistemic states, wherein she (from left to right) implicitly knows q, knows $\neg q$, or does not know whether q, are observationally indistinguishable for the agent: they are p-awareness bisimilar. A p-awareness bisimulation between (e.g.) the left and the right picture is $\mathfrak{R} = \{(s, s''), (t, t''), (t, t''')\}$.

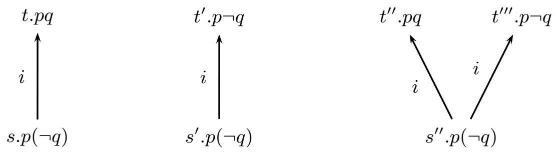

Fig. 1. Agent i is aware of p but unaware of q in state s

3 Language and Semantics

We augment multi-agent epistemic logic with three new operators: $A_i\varphi$, to mean that agent i *is* aware of all the propositional variables in φ; and $A_i^+ p\varphi$ for agent i *becoming aware* of propositional variable p, after which φ is true. The construct $K_i\varphi$, "agent i knows φ" stands in our case for "agent i *intrinsically* knows φ"—the meaning of intrinsic will be explained later.

Definition 3 (Language). *Given are a countably infinite set of propositional variables (facts/atoms) P, and a (disjoint) countably infinite set of agents N. The language $\mathcal{L}$ of individual local awareness is defined as*

$$\varphi ::= \top \mid p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K_i\varphi \mid A_i\varphi \mid A_i^+ p\varphi$$

where $i \in N$ and $p \in P$. Implication $\rightarrow$, disjunction $\vee$, and equivalence $\leftrightarrow$ are defined by abbreviation. For $\neg K_i \neg\varphi$ we write $L_i\varphi$.

The semantics of the awareness operator A_i is purely syntax-based, namely using the *free variables* of a formula. These are defined as: $v(\top) = \emptyset$, $v(p) = \{p\}$, $v(\varphi \wedge \psi) = v(\varphi) \cup v(\psi)$, $v(\neg\varphi) = v(\varphi)$, $v(K_i\varphi) = v(\varphi)$, $v(A_i\varphi) = v(\varphi)$, and $v(A_i^+ p\varphi) = v(\varphi) \setminus \{p\}$. Note that $v(\varphi) \subseteq P$. We explicitly include $\top$ in the language, as the usual abbreviation $p \vee \neg p$ complicates cases where not all agents are aware of p (an agent unaware of p would then not explicitly know truth).

Definition 4 (Semantics). *Let $M = (S, R, \mathcal{A}, V)$ be given.*

$$\begin{array}{ll}
(M,s) \models \top & \\
(M,s) \models p & \textit{iff } s \in V(p) \\
(M,s) \models \varphi \wedge \psi & \textit{iff } (M,s) \models \varphi \textit{ and } (M,s) \models \psi \\
(M,s) \models \neg\varphi & \textit{iff } (M,s) \not\models \varphi \\
(M,s) \models K_i\varphi & \textit{iff } \forall t \in sR_i,\ \forall (M',t') \leftrightarrows_{\mathcal{A}_i(s)} (M,t), (M',t') \models \varphi \\
(M,s) \models A_i\varphi & \textit{iff } v(\varphi) \subseteq \mathcal{A}_i(s) \\
(M,s) \models A_i^+ p\varphi & \textit{iff } (M^{i \mapsto p}, s) \models \varphi
\end{array}$$

where $M^{i \mapsto p} = (S, R, \mathcal{A} \cup \{(i,(t,p)) \mid t \in S\}, V)$. The set of validities (and the logic) is called DLILA (Dynamic Logic of Individual Local Awareness).

Intrinsic knowledge. The treatment of knowledge in this semantics is novel. An agent knows φ only if in all accessible states φ remains true for *every* possible interpretation of all propositional variables that she is unaware of. We achieve this by composing the accessibility relation for an agent with the bisimulation relation modulo the propositional variables of which the agent is unaware. Because the constraints in this composition are interdependent, we have *one* K_i operator in the logical language and not, instead, *two* independent operators, one for standard modal accessibility and another one for bisimulation quantification. If the agent is aware of every propositional variable in the formula φ, the interpretation of knowledge is as for epistemic logic.

Awareness dynamics. Compared to knowledge, the semantics of becoming aware is simple. The complexity of becoming aware can only be seen in the context of intrinsic knowledge. Suppose that the agent is unaware of p and that p is true in all accessible states. We then have that $A_i^+ pK_ip$ is true: after the agent becomes aware of p, p is true. But although the agent considers that as a possibility, she does not know that, and she also considers it possible that after becoming aware of p, she knows that p is false, or that she is uncertain about p: all true are $L_iA_i^+ pK_ip$, $L_iA_i^+ pK_i\neg p$, and $A_i^+ p\neg(K_ip \vee K_i\neg p)$.

In this paper, we made one of three possible choices for awareness dynamics. All three consist of making an unaware variable into an aware variable, i.e., changing the set $\mathcal{A}$ in a model but leaving all other parameters the same. Given state s, one can make agent i aware of the propositional variable p:

- in the actual state (only): $\mathcal{A} \cup \{(i,(s,p))\}$.
- in the actual state and all states accessible for agent i: $\mathcal{A} \cup \{(i,(s,p))\} \cup \{(i,(t,p)) \mid t \in S \text{ and } R_i(s,t)\}$.
- in all states of the model: $\mathcal{A} \cup \{(i,(t,p)) \mid t \in S\}$.

All three are bisimulation invariant (with for the 'actual state only' version the restriction that the operation is performed on a bisimulation contraction, this requires a further adjustment of the definition). You might see the public version of becoming aware as the 'public announcement' version of awareness dynamics: just as in information dynamics, more complex dynamics have more complex axiomatizations, and this is on our future agenda.

KD45 and S5 Apart from the logic $DLILA$ we also consider the logics $DLILA_{\mathrm{L}}$, where every modal operator K_i satisfies the axioms of the logic L. Typical choices of L are S5 and KD45. One should be careful to note that this is *not* a simple case of restriction. Restricting the underlying logic to L (for example KD45) means that in interpreting the formula $K_i\varphi$, we may only consider pointed models (M', t') that satisfy the constraints of L (so transitive, serial and euclidean for KD45). The validities of $DLILA_{\mathrm{L}}$ therefore do not necessarily extend those of $DLILA$. And indeed, each axiomatization also poses new problems.

Specific logics require us to vary the semantics of the operator $A_i^+ p$. For example, given awareness introspection and $S5$, the minimal way of becoming aware makes an agent aware of a propositional variable in the current world *and* in every indistinguishable world (the second option, before). In this paper we show completeness for the logic $DLILA_K$ namely for awareness models M where (S^M, R^M) is a tree, and where becoming aware means becoming aware in every world.

Where to put the complexity? An alternative interaction between knowledge and becoming aware is embodied in the following semantics (presented in [17]):

$$\begin{array}{l} (M,s) \models K_i\varphi \quad \text{iff } \forall t \in sR_i,\ (M,t) \models \varphi \\ (M,s) \models A_i^+ p\varphi \text{ iff } \exists (N,t) \underline{\leftrightarrow}_{\mathcal{A}_i(s)} (M,s), (N^{i \mapsto p}, t) \models \varphi \end{array}$$

Here, the epistemic operator K_i remains the 'classical' one, whereas the becoming aware operator $A_i^+ p$ is the complex one. The advantage is obvious: the novel operator is the only addition to a well-known logic (namely that of [1]). The disadvantage is that a propositional variable may change its value in the process of the agent becoming aware of it; p may be true, but in the transition to a $(P \setminus \{p\})$-bisimilar state it may become false. So, e.g., the agent may become aware that she knows p to be false, even if prior to that she 'implicitly knew' p to be true. In that semantics, K_i does not mean implicit knowledge at all.

It seemed better to stock all the factual change into the mind of the agent only, as in the complex K_i operator, such that the becoming aware operation is merely revealing the veil of incompleteness. For $KD45$ and $S5$ structures that also satisfy 'awareness introspection' the distinction is immaterial, as the two semantics then are identical with respect to explicit knowledge. So, from an agent's point of view, there is no difference.

Proposition 2. *The semantics of $DLILA_K$ are invariant to bisimulation.*

Proof. This is straightforward because relation $\leftrightarrows_A$ is closed under bisimulation.

Examples

1. The introductory example about coffee and wine can be explained by seeing the model on the left as the *equivalence class modulo unawareness of q* of the model on the right. The agent can speculate over all models in that class (cf. the semantics of K_i, with bisimulation except for q). Becoming aware means that a model identical to the right model but with q unaware in all states, is transformed into the right model. On the left, in the actual state where p is false, it is e.g. true that: $\neg A_i q \wedge A_i^+ q \neg K_i q$.
2. Consider again Figure 1, and the roots of the models. In all three cases agent i knows that p. But she does not know in state s that q, because accessible state t is p awareness bisimilar to (e.g.) t' wherein q is false. After becoming aware of q in state s, she knows q: then, any state that is $\{p, q\}$ awareness bisimilar to t *must* satisfy q. So $A_i^+ q K_i q$ is true. Consider a $KD45$ extension of these models, i.e., add access (t, t) on the left, (t', t') in the middle, and $(t'', t''), (t'', t'''), (t''', t''), (t''', t''')$ on the right. Now we have that the agent considers it possible that: after becoming aware of q, she knows that q, or she knows that $\neg q$, or she does not know whether q.
3. Consider the case of $DLILA_{KD45}$, where every agent's accessibility relation is transitive, serial and euclidean. Crucially, in $KD45$, strong beliefs may be mistaken, but you do not consider that possible: to yourself, your beliefs appear knowledge. So $L_i(\neg p \wedge K_i p)$ is inconsistent. However, in $DLILA_{KD45}$ it is valid that an agent i considers it possible that she becomes aware of a propositional variable p that is false and that she believes to be true. That is nothing but speculating about becoming aware of false information that you had reason to accept! A validity of the language is $\neg A_i p \rightarrow L_i A_i^+ p(\neg p \wedge K_i p)$. The interpretation of this formula is shown in Figure 2. The crucial aspect is that the pair $(s, t) \in \mathfrak{R}[\emptyset]$ (the dashed line): agent i cannot a priori distinguish the reality of p being true in the believed world from the speculative option that p is false there but believed true. However, after becoming aware of p (in both s and t) this option is out of reach, as $(s, t) \notin \mathfrak{R}[p]$.

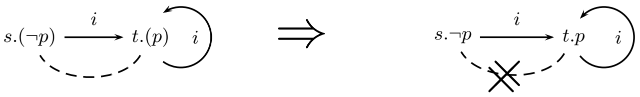

Fig. 2. You can become aware of a false belief

4 Intrinsic, Explicit and Implicit Knowledge

Past literature on knowledge and awareness has focused on the difference between implicit knowledge ("knowing" something without being fully aware of that thing) and explicit knowledge ("knowing" something as well as being fully aware of that thing). Intrinsic knowledge is strictly weaker than explicit knowledge and strictly stronger than implicit knowledge. It allows us to reason about the process of becoming aware, and that is our reason to complicate the existing picture. Implicit knowledge and explicit knowledge are definable in our framework, and we can compare those definitions with the traditional definitions.

Definition 5 (Explicit knowledge K_i^E and implicit knowledge K_i^I)

- $K_i^E \varphi$ *iff* $A_i \varphi \wedge K_i \varphi$ *(explicit knowledge)*
- $K_i^I \varphi$ *iff* $A_i^+ v(\varphi) K_i \varphi$ *(implicit knowledge)*

Expression $A_i^+ v(\varphi)$ means 'becoming aware of a finite set of propositional variables' and is defined in the obvious way. We also have that $K_i^I \varphi$ is equivalent to $A_i^+ v(\varphi) K_i^E \varphi$. The [1] definitions (in bold) are that $(M, s) \models \mathbf{K}_i^I \varphi$ iff $\forall t \in sR_i$, $(M, t) \models \varphi$, and that $(M, s) \models \mathbf{K}_i^E \varphi$ iff $(M, s) \models A_i \varphi$ and $\forall t \in sR_i$, $(M, t) \models \varphi$. We now observe that $K_i^E \varphi$ iff $\mathbf{K}_i^E \varphi$, and that $\mathbf{K}_i^I \varphi$ implies $K_i^I \varphi$ but not vice versa (e.g., if i is unaware of p, but j is aware of p, then i implicitly knows j to know that i is aware of p: $K_i^I K_j^E A_i p$ — this may come closer to implicit knowledge as in [12]). Intrinsic knowledge is clearly not definable in terms of implicit and explicit knowledge, given its semantics employing bisimulation quantification! Interaction between the three kinds of knowledge includes:

Proposition 3. $\models K_i^E \varphi \rightarrow K_i \varphi$ *and* $\models K_i \varphi \rightarrow K_i^I \varphi$.

On the other hand, $\not\models K_i^I \varphi \rightarrow K_i \varphi$. For example, you can implicitly know that p but, as you are unaware of p, you do not intrinsically know that p.

Proposition 4. *Awareness bisimilar states satisfy the same explicit knowledge: If* $(M, s) \models K_i^E \varphi$ *and* $(M, s) \leftrightarroweq_{\mathcal{A}_i(s)} (M', s')$, *then* $(M', s') \models K_i^E \varphi$.

Proof. Note that $A_i \varphi$ means $v(\varphi) \subseteq \mathcal{A}_i(s)$. In the language restricted to $\mathcal{A}_i(s)$ the epistemic awareness states (M, s) and (M', s') are therefore bisimilar in the standard sense, from which follows logical equivalence, thus equivalence of $A_i \varphi \wedge K_i \varphi$ in both states.

5 Decidability

In this section we show decidability via an embedding into bisimulation-quantified modal logics [4]. Bisimulation-quantified modal logic is an extension of multi-modal (such as multi-agent) modal logic with the bisimulation quantifier, $\exists p \varphi$, which is interpreted as: "there is some model bisimilar to the current model except for the atom p, and in which φ is true". We recall the notion of restricted bisimulation already apparent in Definition 2. These logics are interpreted on models without the awareness function but that are otherwise similar.

Definition 6 (Bisimulation Quantified Modal Logic). *Let $L_{\mathcal{C}}$ be the set of validities for a model class $\mathcal{C}$. We define the bisimulation-quantified extension of $L_{\mathcal{C}}$ to be $QL_{\mathcal{C}}$ with the syntax:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [i]\varphi \mid \exists p\varphi$$

where $p \in P$ and $i \in N$, and with the crucial semantic clause:

$$M, s \models_{\mathcal{C}} \exists p\varphi \iff \text{for some } M', t \in \mathcal{C},\ M, s \underline{\leftrightarrow}_{P-p} M', t \text{ and } M', t \models_{\mathcal{C}} \varphi.$$

It is shown in [4] that bisimulation quantified logics are decidable where $L_{\mathcal{C}}$ is an *idempotent transduction logic*; multi-agent **S5** and multi-agent **K** describe such idempotent transduction logics, and consequently have decidable bisimulation quantified extensions. We will give the embedding for $DLILA_K$ in QL_K (where K is the class of models satisfying all **K** validities).

Definition 7. *Let $\varphi \in \mathcal{L}$, and for every agent $i \in N$ and for every propositional variable (atom) $p \in v(\varphi)$, let a_p^i be an atom not appearing in φ, (referred to as an* awareness atom*, where $\mathcal{A}^\varphi$ is the set of awareness atoms for φ). The* embedding *of $DLILA_K$* into *QL_K is given by the recursive function $\psi|_\varphi$ such that:*

$$\begin{array}{rlrl} \top|_\varphi = \top & & (A_i\psi)|_\varphi = \bigwedge_{p \in v(\psi)} a_p^i & \\ p|_\varphi = p & & (K_i\psi)|_\varphi = \bigwedge_{C \subseteq v(\psi)} (A_iC \rightarrow [i]\forall\overline{C^\varphi}\psi|_\varphi) & \\ (\neg\psi)|_\varphi = \neg(\psi|_\varphi) & & (A_i^+ p\psi)|_\varphi = \psi|_\varphi[\top\backslash a_p^i] & \\ (\psi \wedge \chi)|_\varphi = (\psi|_\varphi) \wedge (\chi|_\varphi) & & & \end{array}$$

where $\forall C$ is an abbreviation for $\forall p_0 ... \forall p_n$ for the set of atoms $C = \{p_0, ..., p_n\}$; C^φ is the set of all atoms in C along with the awareness atoms a_p^i where p appears in C; $\overline{C}$ is the complement of C with respect to the set of atoms in φ and the set $\mathcal{A}^\varphi$; and for $C \subseteq v(\varphi)$, A_iC is an abbreviation for $\bigwedge_{p \in C}[i]a_p^i \wedge \bigwedge_{p \in v(\varphi)\backslash C}[i]\neg a_p^i$. Also, $\psi[\chi\backslash p]$ is an abbreviation for the replacement of every free occurrence of the atom p in ψ with the formula χ.

This embedding is a direct encoding of the semantics for $DLILA_K$ into the logic QL_K. The only non-trivial recursions are for $K_i\psi$, which uses bisimulation quantifiers to encode an awareness bisimulation, and $A_i^+ p\psi$, that simply sets agent i's awareness of p to true at every state.

Proposition 5. *Let $M = (S, R, \mathcal{A}, V)$ be a model such that for all atoms $p \in v(\varphi)$, and for all agents $i \in N$, for all $s \in S$, we have $s \in V(a_p^i)$ if and only if $p \in \mathcal{A}_i(s)$. Then for all $s \in S$: $M, s \models \varphi \iff M, s \models \varphi|_\varphi$. (On the right hand side, ignore the awareness parameter of the model in order to interpret $\varphi|_\varphi$.)*

Proof (Sketch). We give this proof by induction over the complexity of formulas. The induction hypothesis holds for $\psi \subset \varphi$ if and only if for all models N where the awareness atoms match the agents' awareness in M we have for every $s \in S^N$, $N, s \models \psi$ if and only if $N, s \models \psi|_\psi$. The base of the induction is the propositional atoms of φ and the truth symbol, $\top$, and these may be seen to support the induction hypothesis. The inductive cases for $\neg\alpha$ and $\alpha \wedge \beta$ follow directly from their semantic definitions, and the case for $A_i\alpha$ follows immediately from the fact that the awareness atoms agree with the agent's awareness.

For $K_i\alpha$, suppose that (N, s) is a pointed model such that $N, s \models K_i\alpha$. Let C be the set of propositional variables $\mathcal{A}_i^N(s) \cap v(\varphi)$. We may proceed as follows:

$$\begin{array}{ll} & N, s \models K_i\alpha \\ \Leftrightarrow [Def] & \forall t \in sR_i^N, \forall (N', t') \underline{\leftrightarrows}_C (N, t),\ N', t' \models \alpha \\ \Leftrightarrow [I.H.] & \forall t \in sR_i^N, \forall (N', t') \underline{\leftrightarrows}_C (N, t),\ N', t' \models \alpha|_\varphi \\ \Leftrightarrow [*] & \forall t \in sR_i^N, \forall (N', t') \cong_{C^\varphi} (N, t),\ N', t' \models \alpha|_\varphi \\ \Leftrightarrow [Def] & \forall t \in sR_i^N, N, t \models \forall \overline{C^\varphi} \alpha|_\varphi \\ \Leftrightarrow [Def] & N, s \models [i] \forall \overline{C^\varphi} \alpha|_\varphi \end{array}$$

The equivalences here are all straightforward, except for the one labelled $*$. In the forward direction this is trivial: every C^φ-bisimulation *is* a C-awareness bisimulation. In the reverse direction, we must note the construction of $\alpha|_\varphi$. Here modalities $[i]$ only appear in the form $\bigwedge_{C \subseteq v(\alpha)} (A_i c \rightarrow [i] \forall \overline{C^\varphi} \alpha|_\varphi)$. With respect to this form we can see that C^φ-bisimulations and C-awareness-bisimulations indeed are equivalent since every application of a modality extends the bisimulation according to awareness function at that point. Since C is defined to be the set of atoms of which agent i is aware, the result follows. Finally, we note that for the interpretation of $A_i^+ p\alpha$ it is enough to manually fix the interpretation of every free occurrence of the atom a_i^p in $\alpha|_\varphi$ to true, as $A_i p$ will be true in every state.

Thus we have a translation from $DLILA_K$ to the bisimulation quantified logic QL_K that preserves the meaning of formulas (given a set of awareness atoms in the model). Decidable satisfiability and model-checking follow. We have not yet investigated the lower bound for complexity of the translation. The translation given is quite general, as it also suffices for QL_{KD45} or QL_{S5}.

6 Axiomatization

We provide an axiomatization **DLILA** for $DLILA_K$, and we show it to be sound and complete. The propositional rules and axioms, and those for knowledge (only involving K_i), are standard. The axioms for awareness (for A_i) simply capture the syntactic definition. The interaction between knowledge and awareness is governed by the axioms **AK1**–**AK4**.

C0	All tautologies of prop. logic	**K**	$K_i(\varphi \rightarrow \psi) \rightarrow K_i\varphi \rightarrow K_i\psi$
MP	From φ and $\varphi \rightarrow \psi$ infer ψ	**Nec**	From φ infer $K_i\varphi$
A1	$A_i(\varphi \wedge \psi) \leftrightarrow A_i\varphi \wedge A_i\psi$	**A2**	$A_i\neg\varphi \leftrightarrow A_i\varphi$
A3	$A_i K_j \varphi \leftrightarrow A_i\varphi$	**A4**	$A_i A_j \varphi \leftrightarrow A_i\varphi$
A5	$A_i A_j^+ p\varphi \leftrightarrow A_i p \wedge A_i\varphi$	**A6**	$A_i\top$

AK1 $K_i\varphi \wedge \neg A_i p \rightarrow K_i\varphi[\psi \backslash p]$
AK2 From $A_i\varphi \wedge K_i\varphi \rightarrow K_i\psi$ infer $A_i\psi \wedge K_i\varphi \rightarrow K_i\psi$
AK3 $(K_i(p \rightarrow \varphi) \vee K_i(\neg p \rightarrow \varphi)) \wedge \neg A_i p \rightarrow K_i\varphi$
AK4 $(K_i(A_j p \rightarrow \varphi) \vee K_i(\neg A_j p \rightarrow \varphi)) \wedge \neg A_i p \rightarrow K_i\varphi$.

In **AK1** we require that ψ is free for p in φ, and the axioms **AK3** and **AK4** may only be applied in the case where the atom p does not appear outside the scope of a modal (knowledge) operator. Axioms **AK1** and **AK2** are not required in the completeness proof, but we have left them in as they represent important principles that hold in *all* semantic variations of $DLILA$: **AK1** shows that if an agent is not aware of an atom, then the agent may not distinguish the interpretation of that atom from the interpretation of an arbitrary proposition; **AK2** states that if intrinsic knowledge of ψ can be derived from explicit knowledge of φ, then intrinsic knowledge of ψ may also be derived from intrinsic knowledge of φ and awareness of ψ. Axioms **AK3** and **AK4** are specific to the **K** semantics: they capture the intrinsic nature of the knowledge operator: if an agent is unaware of an atom, he does not refute any interpretation of that atom, nor does he refute the interpretation of any agent's awareness of that atom.

Finally we present axioms for becoming aware. We note from the semantics that if an agent i becomes aware of an atom this will only affect the interpretation for formulas $A_i\varphi$ or $K_i\varphi$. Consequently A_i^+p commutes with all other operators.

B0 $A_i^+p\top$ **B1** $A_i^+pq \leftrightarrow q$
B2 $A_i^+p(\varphi \wedge \psi) \leftrightarrow A_i^+p\varphi \wedge A_i^+p\psi$ **B3** $A_i^+p\neg\varphi \leftrightarrow \neg A_i^+p\varphi$
B4a $A_jp \rightarrow (A_i^+pK_j\varphi \leftrightarrow K_jA_i^+p\varphi)$ **B4b** $K_iA_i^+p\varphi \rightarrow A_i^+pK_i\varphi$
B5a $A_i^+pA_j\varphi \leftrightarrow A_j\varphi$ where $i \neq j$ **B5b** $A_i^+pA_i\varphi \leftrightarrow A_i\varphi[\top\backslash p]$
B6 $A_i^+pA_j^+q\varphi \leftrightarrow A_j^+qA_i^+p\varphi$

Soundness and completeness. The soundness is straightforward. We show completeness for **DLILA** by constructing a canonical model for any formula using maximal consistent sets of formulas in $\mathcal{L}$—proofs are in the appendix.

Definition 8. *The* canonical model *is built from the set $\mathcal{S}$ of all maximal consistent sets of formulas with respect to the system* **DLILA**. *Further we define $\mathcal{M} = (\mathcal{S}, \mathcal{R}, \mathcal{A}, \mathcal{V})$ where:*

- *for all $i \in N$, for all maximal consistent sets $\sigma, \tau \in \mathcal{S}$, $(\sigma, \tau) \in \mathcal{R}_i$ if and only if for all formulas $A_i^+v(\varphi)K_i\varphi \in \sigma$, we have $\varphi \in \tau$;*
- *for all $\sigma \in \mathcal{S}$ for all $i \in N$, for all $p \in P$, we have $p \in \mathcal{A}_i(\sigma)$ if and only if $A_ip \in \sigma$;*
- *for all $\sigma \in \mathcal{S}$, for all $p \in P$, we have $\sigma \in \mathcal{V}(p)$ if and only if $p \in \sigma$.*

Proposition 6. *Every* canonical model *is an epistemic awareness model.*

Proof. In the presence of complete awareness, intrinsic knowledge is equivalent to explicit knowledge, and the logic of explicit knowledge is canonical. We note that the awareness function $\mathcal{A}$ is constant for each agent's local state because of the axiom **AK3**.

Lemma 1 (Truth Lemma). *For every $\sigma \in \mathcal{S}$, for every formula φ, we have $\varphi \in \sigma$ if and only if $\mathcal{M}, \sigma \models \varphi$.*

(Proof in Appendix.) It follows that for every consistent formula φ we may construct a model so the axiomatization **DLILA** is complete for the logic $DLILA$.

7 Comparison

Our approach is in some respects simpler and more constrained than [8]. From the epistemic awareness structure we are able to implicitly derive a complete lattice of spaces via awareness bisimulation, whereas in [8] this structure is given explicitly. In other words, we have a succinct, technical tool to derive that result.

The principles $A1, \ldots, A6$ in **DLILA** straightforwardly correspond to (a multi-agent version of) L^{KXA} in [6] and Proposition 3 in [8]—epistemic operators K_i in the scope of awareness operators can be replaced by the *explicit* knowledge operators K_i^E assumed by those authors; $A5$ is a 'mix' axiom relating to dynamics. Principles $AK1$ and $AK2$ were conceived using results for bisimulation-quantified logics and are strictly about *intrinsic* knowledge only.

Although we do not explicitly have propositional quantifiers, they are indirectly present in intrinsic knowledge operators. Propositional quantification is integrated with awareness and knowledge in [7] (and in various precursors). This concerns quantification over the set of formulas of which an agent is aware. They interestingly mention that "Using semantic valuations [for quantification] does not work in the presence of awareness" [7, p.506]; although of course correct, we are wondering if our work may make the authors reconsider the suggested scope of that remark.

Dynamics of (factual) awareness is presented in [9,5,19]. In [9] becoming aware means (initially) becoming ignorant about that proposition. It uses an algebraic approach. Becoming ignorant is also the approach in the recent [19], that contains various other novelties. In [5], the approach in Section 3 is similarly dynamic modal as ours, and it provides an integrated combination of syntactic and semantic awareness.

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Appendix: Proof of Truth Lemma 1

For a convenient proof, we give a syntactic version of awareness bisimulation.

Definition 9. *We say a formula of DLILA is* explicit *if it is built from the following syntax:* $\varphi ::= \top \mid p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K_i\varphi \wedge A_i\varphi \mid A_i\varphi$. *For every* $C \subseteq P$, *let* $\mathcal{B}(C)$ *be a binary relation on* $\mathcal{S}$ *satisfying for all* $\sigma, \tau \in \mathcal{S}$, $(\sigma, \tau) \in \mathcal{B}(C)$ *if and only if for every explicit formula* φ *containing only the atoms in* C, *we have we have* $\varphi \in \sigma$ *implies* $\varphi \in \tau$. *We refer to such formulas* φ *as* C*-explicit.*

The following lemma is a strengthening of Proposition 4 and shows the correspondence between Definition 9 and Definition 2.

Lemma 2. *For every* $\sigma, \tau \in \mathcal{S}$, *for every* $C \subseteq P$ *we have* $(\mathcal{M}, \sigma) \leftrightarroweq_C (\mathcal{M}, \tau)$ *if and only if* $(\sigma, \tau) \in \mathcal{B}(C)$.

Proof (Sketch). ($\Longrightarrow$) We show by induction over the complexity of formulas that for any σ, τ where $(\mathcal{M}, \sigma) \leftrightarroweq_C (\mathcal{M}, \tau)$, we have, for any C-explicit formula φ, $\varphi \in \sigma$ if and only if $\varphi \in \tau$. In the case that φ is a propositional atom or $\top$ it is clear that $\varphi \in \sigma$ iff $\varphi \in V(\sigma)$ iff $\varphi \in V(\tau)$ iff $\varphi \in \tau$. The inductions for the propositional operators $\wedge$ and $\neg$ are similarly straightforward. For $A_i\varphi$, by application of the axioms **A1-A6** we have $A_i\varphi \in \sigma$ iff for all atoms p in φ we have $A_ip \in \tau$. By the **aware** clause of Definition 2, this is equivalent to $A_ip \in \tau$ so we must have $A_i\varphi \in \tau$. Finally, if $K_i\varphi \wedge A_i\varphi \in \sigma$, then for all $\sigma' \in \sigma\mathcal{R}_i$ we have $\varphi \in \sigma'$. By Definition 2 for all $\tau' \in \tau\mathcal{R}_i$, there exists $\sigma' \in \sigma\mathcal{R}_i$ such that $\tau' \underline{\leftrightarrow}_{C\cap\mathcal{A}_i(\sigma)} \sigma'$. By the induction hypothesis it follows that $\varphi \in \tau'$, so $K_i\varphi \wedge A_i\varphi \in \sigma$ iff $K_i\varphi \wedge A_i\varphi \in \tau$.

($\Longleftarrow$) Here we show that the relations $\mathcal{B}(C)$ satisfy the properties specified in Definition 2. Clearly the clauses **atom** and **aware** hold since if $(\sigma, \tau) \in \mathcal{B}(C)$ then σ and τ agree on all C-explicit formulas which includes the atoms in C, and the awareness of those atoms. To see **forth** holds, suppose that $(\sigma, \tau) \in \mathcal{B}(C)$. Then for all agents i, for all $\sigma' \in \sigma\mathcal{R}_i$, for all $C \cap \mathcal{A}_i(\sigma)$-explicit formulas $\varphi \in \sigma'$, we have $L_i\varphi \wedge A_i\varphi \in \sigma$. By the definition of $\mathcal{B}(C)$ we have, for all $C \cap \mathcal{A}_i(\sigma)$ formulas $\varphi \in \sigma'$, $L_i\varphi \wedge A_i\varphi \in \tau$. By the axiom **B4a** every finite subset of the $C \cap \mathcal{A}_i(\sigma)$-explicit formulas in σ' is consistent with the set of implicit knowledge formulas, $\{\psi \mid A^+v(\psi)K_i\psi \in \tau\}$. As there is no finite proof of inconsistency we may conclude that the set of $C \cap \mathcal{A}_i(\sigma)$-explicit formulas in σ' is consistent with the set of implicit knowledge formulas in τ. By Definition 8 there is some $\tau' \in \tau\mathcal{R}_i$ such that $(\sigma, \tau') \in \mathcal{B}(C \cap \mathcal{A}_i(\sigma)$, as required. The case for **back** is handled symmetrically.

Lemma 2 provides a compelling justification for the notion of awareness bisimulation. Two states are C-awareness bisimilar exactly when they agree on all C-explicit formulas. We continue with the proof of the Truth Lemma 1 proper.

Proof (Sketch). This lemma is given by induction over the complexity of formulas. The base case, where $\varphi \in P$ or $\varphi = \top$ is a direct application of the definition of $\mathcal{V}$, so we may assume for all $\psi \subset \varphi$, for all $\sigma \in \mathcal{S}$ we have $\psi \in \sigma$ if and only if $\mathcal{M}, \sigma \models \psi$. The induction proceeds as follows:

$\neg$ Suppose $\varphi = \neg\psi$. Then since $\psi \in \sigma$ if and only if $\mathcal{M}, \sigma \models \psi$, from the consistency of σ we have $\varphi \in \sigma$ if and only if $\mathcal{M}, \sigma \models \varphi$.

$\wedge$ Suppose $\varphi = \psi_1 \wedge \psi_2$. Then since $\psi_i \in \sigma$ if and only if $\mathcal{M}, \sigma \models \psi_i$, from the consistency of σ we have $\varphi \in \sigma$ if and only if $\mathcal{M}, \sigma \models \varphi$.

A_i Suppose $\varphi = A_i\psi$. Then clearly by the axioms **A1-A7**, $A_i\psi \in \sigma$ if and only if, for all atomic propositions $p \in \psi$ we have $A_ip \in \sigma$. This is equivalent to $(\mathcal{M}, \sigma) \models A_ip$ for all atoms, p in ψ, which is equivalent to $(\mathcal{M}, \sigma) \models A_i\psi$.

K_i Suppose $L_i\psi \in \sigma$. Let $\Psi = \{\alpha \mid K_i\alpha \wedge A_i\alpha \in \sigma\}$. Now $\Psi \cup \{\psi\}$ is consistent (since the conjunction γ of every finite subset appears in σ as $L_i\gamma$), and furthermore, it must be consistent with the $\mathcal{A}_i(\sigma)$-explicit formulas that appear in some $\tau \in \sigma\mathcal{R}_i$. Therefore, we may find a maximal consistent set τ' such that $\psi \in \tau' \underline{\leftrightarrow}_{\mathcal{A}_i(\sigma)} \tau \in \sigma\mathcal{R}_i$, so $(\mathcal{M}, \sigma) \models L_i\psi$ as required.

Conversely, suppose that $(\mathcal{M}, \sigma) \models L_i\psi$. We proceed by induction over the knowledge-depth of ψ, where the induction hypothesis is, that for all ψ of knowledge depth n:

1. for all $\tau \in \mathcal{S}$, $\vdash L_i(\tau_\psi^\Gamma) \wedge \overline{A_i\Gamma}^\psi \rightarrow L_i\tau_\psi$, and
2. for all $\tau \in \mathcal{S}$, $(\mathcal{M}, \tau) \models \psi$ if and only if $\psi \in \tau$.

Here, Γ is a set of propositional variables; τ_ψ^Γ is the set of subformulas of ψ containing only atoms from Γ that appear in the set τ; $\overline{A_i\Gamma}^\psi$ is an abbreviation for $\bigwedge\{\neg A_i p \mid p \in \psi\backslash\Gamma\}$; and τ_ψ is the set of subformulas of ψ that appear in τ. This is sufficient to show $\mathcal{M},\sigma) \models L_i\psi$ implies $L_i\psi \in \sigma$ since if $(\mathcal{M},\sigma) \models L_i\psi$, we have $\psi \in \tau$ for some $\tau \underline{\leftrightarrow}_{A_i(\sigma)} \tau' \in \sigma\mathcal{R}_i$. By Lemma 2, $\tau_\psi^{A_i(\sigma)} \in \tau'$, and hence $L_i(\tau_\psi^{A_i(\sigma)}) \in \sigma$. Applying the inductive hypothesis we have $L_i\psi \in \sigma$ as required.

For the base case, iff ψ has knowledge depth 0, we can see from axioms **B0-B7** and **A1-A6** that ψ is effectively a propositional formula where the atoms are either propositional atoms, or agents' awareness of propositional atoms. Now there are two cases: if $A_i\psi \in \sigma$, then by the axiom **B4b** we have for every $\tau \in \sigma\mathcal{R}_i$, for every $\tau' \underline{\leftrightarrow}_{\mathcal{A}_i(\sigma)} \tau$, $\psi \in \tau'$, so $(\mathcal{M},\sigma) \models K_i\psi$ and we are done. Alternatively, if $A_i\psi \notin \sigma$, then there are some atoms in ψ that agent i is not aware of at σ. Let $\mathcal{T} = \{p, A_j p \mid p, j \in v(\psi)\}$. For any τ, τ' where $\tau \underline{\leftrightarrow}_{\mathcal{A}_i(\sigma)} \tau' \in \sigma\mathcal{R}_i$ there is a subset of $\mathcal{T}$ true at τ. We may apply the axioms **AK3** and **AK4** to derive

$$L_i\chi(\tau) = L_i\left(\bigwedge_{\alpha\in\mathcal{T}\cap\tau} \alpha \wedge \bigwedge_{\alpha\in\mathcal{T}\backslash\tau} \neg\alpha \wedge \psi\right)$$

Therefore there is some maximal consistent set ρ containing $\chi(\tau)$, and as τ' agrees with ρ on the interpretation of all atoms up to the depth of ψ, we must have $(\mathcal{M},\tau') \models \psi$ as required. As this is the case for every $\tau \in \mathcal{S}$ it must be that $\vdash L_i(\tau_\psi^\Gamma) \wedge \overline{A_i\Gamma}^\psi \rightarrow L_i\tau_\psi$.

For the inductive step we proceed in a similar fashion. Suppose $\tau \underline{\leftrightarrow}_{\mathcal{A}_i}(\sigma)\tau' \in \sigma\mathcal{R}_i$. Given that we may apply the axiom **AK4** to replicate the awareness state of agents at τ'. We may then apply the inductive hypothesis to infer $K_j\psi_k$ at $(\mathcal{M},\tau)$, where ψ_k has knowledge depth less than n. Finally we may again apply **AK3** and **AK4** to replicate the interpretation of atoms, and other agents' awareness of the atoms at τ. As ψ may be written as a Boolean combination of atoms, agents' awareness of atoms and formulas $K_j\psi_k$ (where the knowledge depth of ψ_k is less than n), the result follows.

$A_i^+ p$ Suppose that $\varphi = A_i^+ p\psi$, and $\varphi \in \sigma$. From axiom **B3** we can see the set $\tau = \{\alpha \mid A_i^+ p\alpha \in \sigma\}$ is maximal and consistent. Furthermore from Definition 8 we can see that for all j, for every $\rho \in \mathcal{S}$ we have $\rho \in \sigma\mathcal{R}_j$ if and only if $\rho \in \tau\mathcal{R}_j$. Thus the successors of τ are exactly the successors of σ. Also, from the axioms **B0-B6** we have for all atoms p, $\tau \in \mathcal{V}(p)$ iff $\sigma \in \mathcal{V}(p)$, for all agents $j \neq i$ we have $\mathcal{A}_j(\sigma) = \mathcal{A}_j(\tau)$ and finally $\mathcal{A}_i(\sigma) \cup \{p\} = \mathcal{A}_i(\tau)$. From this we can see $(\mathcal{M},\tau)$ is bisimilar to $(\widehat{\mathcal{M}}^{i\mapsto p},\sigma)$, where $\widehat{\mathcal{M}}$ is the tree unwinding of $\mathcal{M}$. As $\psi \in \tau$, the result follows.

Conversely, suppose that $(\mathcal{M},\sigma) \models A_i^+ p\psi$ and for contradiction, suppose that $\neg A_i^+ p\psi \in \sigma$. Applying **B3** we have $A_i^+ p\neg\psi \in \sigma$. From the argument above it follows that a model bisimilar to $(\widehat{\mathcal{M}}^{i\mapsto p},\sigma)$ satisfies $\neg\psi$, so the contradiction follows from the bisimulation invariance of $DLILA_K$ (Proposition 2).