Knowledge actions in games and multiagent systems

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(extended abstract)

1 Introduction

The area of dynamic epistemics has come to the attention of the research community by the treatment of public announcements in the ‘Muddy Children Problem’ [FHMV95]. Subsequent work includes adding dynamic operators for actions to a multiagent epistemic language [Ger99, BMS00, vD00, vD01]. Part of the interest of our approach lies in the detailed description of epistemic actions in games. We restrict ourselves to S5 (equivalence) models and states.

In this abstract we give a short overview of our language, including an operator for concurrent actions, and we demonstrate its wide applicability by an overview of action descriptions in multiagent systems and in games.

2 Knowledge actions

Syntax To a standard multiagent epistemic language with common knowledge for a set A of agents and a set P of atoms [MvdH95, FHMV95], we add dynamic modal operators for programs that are called knowledge actions and that describe actions. The language $L_A^P$ and the knowledge actions $KA^-$ are defined by simultaneous induction.

Definition 1 (Dynamic epistemic logic – $L_A^P$) $L_A^P$ is the smallest set such that, if $p \in P, \varphi, \psi \in L_A^P, a \in A, B \subseteq A, \alpha \in KA^-$, then $p, \neg \varphi, (\varphi \land \psi), K_a \varphi, C_B \varphi, [a] \varphi \in L_A^P$.

Definition 2 (Knowledge actions – $KA^-$) Given a set of agents $A$ and a set of atoms $P$, the set of knowledge actions $KA^-$ is the smallest set such that, if $\varphi \in L_A^P, \alpha, \alpha' \in KA^-, B \subseteq A, b \in Bu(\alpha)$, then: $?\varphi, L_B \alpha, (\alpha ; \alpha'), (\alpha \cup \alpha'), (\alpha \cap \alpha') \in KA^-.$

Other propositional connectives and modal operators are defined by abbreviations. Outermost parentheses of formulae or actions are deleted whenever convenient. A program $?\varphi$ is a test, the program constructor $L_B$ is called learning, ‘;’ stands for sequential execution, ‘$?$’ for nondeterministic choice, and ‘$\sqcup$’ for concurrent execution. The knowledge actions $KA$ without superscript, also include a globally applicable operation ‘!’ called local choice, that is less essential to this presentation. Local choice constrains the interpretation of the knowledge action bound by it.

Semantics We have deleted details of the semantics from this extended abstract. The semantics of $L_A^P$ (on equivalence models) is defined as usual [MvdH95], plus an additional clause for the meaning of dynamic operators. The interpretation (called local interpretation) of a dynamic operator is defined as a relation between an equivalence state and a set of equivalence states: $s \models [\alpha] \varphi \iff \forall S' : s[\alpha]S' \Rightarrow \exists s' \in S' : s' \models \varphi$. The interpretation of a concurrent knowledge action is typically a relation between one state and a set of two states.

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For all actions except concurrent knowledge actions it is more intuitive to think of their interpretation as a relation between states than as a relation between a state and a set of states: if \( s[s'] \), we like to think of \( s' \) as the result of executing \( a \) in \( s \). We therefore introduce the helpful abbreviation: \( s[a] \Rightarrow s[s'] \). If \( [a] \) is functional, we write \( s[a] \) for the unique \( s' \) such that \( s[a]s' \). For details and motivation, see [vD00, vD01]. We continue with various applications of this language to dynamics in games and multiagent systems.

3 Nightclub or lecture

Anne and Bert are in a bar, sitting at a table. A messenger comes in and delivers a letter to Anne. The letter contains either an invitation for a night out in Amsterdam, or an obligation to give a lecture instead. Anne and Bert commonly know that these are the only alternatives. This situation can be modelled as follows: There is one atom \( p \), describing ‘the letter contains an invitation for a night out in Amsterdam’, so that \( \neg p \) stands for the lecture obligation. There are two agents 1 (Anne) and 2 (Bert). \textit{Letter} is the model \( \langle \{w, w'\}, \{\sim_1, \sim_2\}, V \rangle \) with both \( \sim_1 \) and \( \sim_2 \) the universal relation on \( \{w, w'\} \), and with \( V(p) = \{w\} \). Now suppose \( p \) is actually the case. This corresponds to the state (Letter, \( w \)). We list four actions that are executable in that state. Figure 1 pictures (some) states resulting from their execution. For simplicity, all worlds in the figure are named by their atomic description, i.e. with either \( p \) or \( \neg p \).

\begin{example}[tell] Anne is invited for a night out in Amsterdam and reads the letter aloud. \end{example}

\begin{example}[read] Bert is seeing that Anne reads the letter (and this is publicly known). \end{example}

\begin{example}[cheat] Bert orders a drink at the bar so that Anne may have looked at the contents of the (unsealed) letter. However, Bert suspects her of having cheated (and this is publicly known). \end{example}

\begin{example}[double] Bert orders a drink at the bar and Anne goes to the bathroom so that they now suspect each other of having looked at the contents of the letter. (We picture just one execution of this action, namely where they both have actually looked at its contents. There are three other executions of this action.) \end{example}
The descriptions of the actions in this language are:

| tell | $L_{12}?p$ |
| read | $L_{12}(L_1?p \cup L_1?p)$ |
| cheat | $L_{12}(L_1?p \cup L_1?p \cup ?\top)$ |
| double | $L_{12}(L_1?p \cup L_1?p \cup ?\top)$; $L_{12}(L_2?p \cup L_2?p \cup ?\top)$ |
|       | $L_{12}(L_1?p \cup L_2?p \cup L_1?p \cup L_1?p \cup L_2?p \cup L_2?p \cup ?\top)$ |

For example, we may paraphrase the description of cheat as: ‘(1 and 2 learn that (1 learns $p$) or (1 learns not $p$) or (nothing happens)))’. The ‘test on truth’ $\top$ always succeeds and therefore stands for ‘nothing happens’. In double, the operator ‘$;$’ stands for sequential execution. The description below it is an alternative description of double. Note that in the first description of double we have arbitrarily chosen to let 1 (possibly) cheat first and 2 second, although we have no reason to assume this order. We omit details of computing the interpretation of tell, read and double.

4 Muddy Children

In [GG97], the dynamics in ‘Muddy Children’ [FHMV95] is presented as a sequence of public announcements of propositions. Public announcement of $\varphi$ is described in $L_\lambda$ as a knowledge action $L_\lambda?\varphi$. For readability, we use this translation.

Assume three children 1, 2, 3 and let $m_i$ stand for ‘i is muddy’. The declaration of ‘father’ that there is at least one muddy child is described by $L_{123}?(m_1 \lor m_2 \lor m_3)$. A round of ‘nobody steps forward’ is analysed as a public announcement of a conjunctive proposition that none of the children knows whether he/she is muddy:

$$L_{123}?(\neg K_1 m_1 \land \neg K_1 \neg m_1) \land (\neg K_2 m_2 \land \neg K_2 \neg m_2) \land (\neg K_3 m_3 \land \neg K_3 \neg m_3))$$

We think that this analysis, although correct, does not take into account the fine structure of this action, where children simultaneously publish that they don’t know whether they are muddy. We prefer an analysis where ‘nobody steps forward’ is an action that is composed of subprograms ‘i does not step forward’, etc. This cannot be described in [GG97], but can be done in $L^{\{m_1, m_2, m_3\}}_{\{1,2,3\}}$:

Example 5 (The fine structure of muddy children) A description in $L^{\{m_1, m_2, m_3\}}_{\{1,2,3\}}$ of ‘nobody steps forward’ in ‘Muddy children’ is:

$$L_{123}(L_{123}?(\neg K_1 m_1 \land \neg K_1 \neg m_1) \cap L_{123}?(\neg K_2 m_2 \land \neg K_2 \neg m_2) \cap L_{123}?(\neg K_3 m_3 \land \neg K_3 \neg m_3))$$

5 Card games

Various card games involve exchange of information where subgroups of different size interact in the communication: the murder game Cluedo (or Clue), the ‘family game’ (where one gathers ‘four of a kind’), the game of memory. We restrict ourselves to modelling moves where cards do not change hands. Various moves are of epistemic interest: (only) showing a card to another player, moves that are responses involving choice, simultaneous moves and actions, purely epistemic actions such as ending your move (declaring that you cannot win). Most moves can be illustrated by the simple example of three players each holding one card.

Three players and three cards Three players hold three cards, suppose player 1 holds the red card, 2 holds white and 3 holds blue. The equivalence state ($Heza, rwb$) = ({$rw, rw, rbr, wbr, brw, brw, brow, brow, brow$}, {$\sim_1, \sim_2, \sim_3, V$}, $rw$) represents the knowledge of the players in this state, see figure 2. In deal of cards $ijk$ player 1 holds card $i$, 2 holds $j$ and 3 holds $k$. There are six deals of three cards over three players. The domain of the (hexagonal) model $Heza$ consists of these
deals. Two deals cannot be distinguished from each other by a player if he holds the same card in both. This induces the equivalence relations on the domain, e.g., the partition \( \sim_1 \) for player 1 is \( \{\{\text{rwb}, \text{rbw}\}, \{\text{wbr}, \text{wrb}\}, \{\text{brw}, \text{bur}\}\} \). The following actions can be executed in this state \( (\text{Hexa}, \text{rwb}) \):

**Example 6 (table)** Player 1 puts the red card (face up) on the table.

**Example 7 (show)** Player 1 shows (only) player 2 the red card. Player 3 cannot see the face of the shown card, but notices that a card is being shown.

**Example 8 (whisper)** Player 2 asks player 1 to tell him a card that he (1) doesn’t have. Player 1 whispers in 2’s ear “I don’t have blue”. Player 3 notices that the question is answered, but cannot hear the answer.

We assume that only the truth is told. In show and whisper, we assume that it is publicly known what 3 can and cannot see or hear. Figure 2 pictures the states that result from updating the current game state \( (\text{Hexa}, \text{rwb}) \) with the information contained in the three actions. In table it suffices to eliminate some worlds: after 1’s action, the four deals of cards where 1 does not hold red are eliminated. It is publicly known that they are no longer accessible. This update is a public announcement. In show we cannot eliminate any world. After this action, e.g., 1 can imagine that 3 can imagine that 1 has shown red, but also that 1 has shown white, or blue. However, some links between worlds have now been severed: whatever the actual deal of cards, 2 cannot imagine any alternatives after execution of show. In whisper player 1 can choose whether to say “not white” or “not blue”, and the resulting game state has twice as many (nonsimilar) worlds as the current state, because for each deal of cards this choice can be imagined to have been made.

We can paraphrase some more of the structure of the actions. In table all three players learn that player 1 holds the red card. In show, 1 and 2 learn that 1 holds red, whereas the group consisting of 1, 2 and 3 learns that 1 and 2 learn which card 1 holds, or, in other words: that either 1 and 2 learn that 1 holds red, or that 1 and 2 learn that 1 holds white, or that 1 and 2 learn
that 1 holds blue. The choice made by subgroup \{1,2\} from the three alternatives is local, i.e.
known to them only, because it is hidden from player 3. The description of the example actions
in KA^- is:

<table>
<thead>
<tr>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>table</td>
</tr>
<tr>
<td>show</td>
</tr>
<tr>
<td>whisper</td>
</tr>
</tbody>
</table>

**Unsuccessful updates** In the murder game Cluedo, after having been shown a card, a player
then either explicitly makes a final accusation or implicitly foregoes that opportunity and passes
his move to the next player. This ‘passing one’s move’ turns out to be a real epistemic action.
This will be clear if the ‘suspicions’ (i.e. request for three different cards) uttered gets the following
response: all other players cannot show a card. It then holds that either the murder cards, i.e. the
cards on the table, must be the requested cards, or that the requesting player must hold one of the
requesting cards. By not winning, i.e. not declaring knowledge of the murder cards, the requesting
player therefore implicitly declares having one of the requested cards. Even more interesting are
actions where publishing information results in it being no longer valid:

An **unsuccessful update** ([Ger99]) is a KA^- knowledge action \( L_A?\varphi \), i.e. an announcement
to group A, for which there is a knowledge state \((M,w)\) such that \((M,w)[L_A?\varphi] \not\models \varphi\). Formula
\( \varphi \) is called the unsuccessful update formula.

**Example 9 (Unsuccessful update)** In the initial state of the game for three players and three
cards, with actual deal rwb, player 1 says to other players: “You don’t know that I have the red
card.”. We have to be more precise: player 1 is implicating “I have the red card and both of you
don’t know that.” The action executed is \( L_{123}?K_1(r_1 \land \neg K_2r_1 \land \neg K_3r_1) \). We have that:

\[
(Heza, rwb)[L_{123}?K_1(r_1 \land \neg K_2r_1 \land \neg K_3r_1)] \not\models K_1(r_1 \land \neg K_2r_1 \land \neg K_3r_1)
\]

We are still investigating whether “nobody can win” in Cluedo can be an unsuccessful update,
and whether it could ever occur in an optimal strategy.

**Choosing between cards** A show action in Heza is a forced action: either you have one of
the requested cards, and you show it, or you don’t, and you can’t show any of them. The game
for 3 players and 4 cards is the most simple knowledge game where some player can choose which
card to show. Assume that player 3 holds 2 cards. The cards are called north, east, south, and
west; or n,e,s,w. The actual deal of cards is: 1 holds north, 2 holds east, and 3 holds south and
west: we write nesw. The initial state \((Tetra,nesw)\) of this game is pictured on the left in figure
3. Its shape is that of a truncated tetrahedron.

**Example 10 (Choosing between cards)** The action of player 3 showing one of his cards to
player 2 is described by the KA^- action:

\[ L_{123}(L_{23}?n_3 \cup L_{23}?e_3 \cup L_{23}?s_3 \cup L_{23}?w_3) \]

One of the two possible executions of this action is shown on the right in the figure.\(^1\)

The next example involves concurrency.

**Example 11 (Simultaneously showing cards)** Player 3 shows one card (only) to player 1,
with his left hand, and (simultaneously) the other card (only) to player 2, with his right hand.
This action is described by the knowledge action:

\[ L_{123}(\bigcup_{i\neq j}(L_{13}?i_3 \cap L_{23}?j_3)) \]

where \(i, j\) range over n,e,s,w. In each game state this action has two possible executions: player
3 may choose whether to show 1 his i or his j card, the other card is then necessarily shown to 2.

\(^1\) The action actually executed is \( L_{123}(L_{23}?n_3 \cup L_{23}?e_3 \cup L_{23}?s_3 \cup L_{23}?w_3) \), where \(?\) stands for local choice.
Figure 3: On the left, a truncated tetrahedron representing the initial knowledge state where 1 holds north, 2 holds south, and 3 holds east and west. Access for player 1 is only shown in a typical case. On the right, a truncated octahedron representing the state of the game after 3 has shown 2 his south card, given a request for one of his cards. Again, not all access for player 1 is shown.

6 Conclusions

Other examples, such as describing the suspicion by group A of an action α taking place as \( L_A(\alpha\cup\top) \), and an extended example involving the distribution of information over a (telephone) network (‘spreading gossip’), have been deleted from this abstract.

We have presented various uses of a dynamic epistemic language. We expect it can be applied to many more problems of interest in game theory and multiagent systems.

References


