‘Knowable’ as ‘known after an announcement’

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Abstract

Public announcement logic is an extension of multi-agent epistemic logic with dynamic operators to model the informational consequences of announcements to the entire group of agents. We propose an extension of public announcement logic with a dynamic modal operator that expresses what is true after *any* announcement: $\Diamond\varphi$ expresses that there is a truthful announcement ψ after which φ is true. This logic gives a perspective on Fitch's knowability issues: for which formulas φ does it hold that $\varphi \rightarrow \Diamond K\varphi$? We give various semantic results, and we show completeness for a Hilbert-style axiomatization of this logic. There is a natural generalization to a logic for arbitrary events.

1 Introduction

One motivation to formalize the dynamics of knowledge is to characterize how truth or knowledge conditions can be realized by new information. From that perspective, it seems unfortunate that in public announcement logic [23, 13, 31] a true formula may become false because it is announced. The prime example is the Moore-sentence ‘atom p is true and you do not know that’, formalized by $p \wedge \neg Kp$ [20, 15], but there are many other examples [29]. After the Moore-sentence is announced, you know that p is true, so $p \wedge \neg Kp$ is now false. This is formalized as $\langle p \wedge \neg Kp\rangle Kp$, and $\langle p \wedge \neg Kp\rangle \neg(p \wedge \neg Kp)$, respectively. The part ‘$\langle p \wedge \neg Kp\rangle$’ is a diamond-style dynamic operator representing the announcement. Therefore, the way to make something known may not necessarily be to announce it. Is there a different way to get to know something?

The realization of knowledge (or truth) by new information can be seen as a specific form of what is called ‘knowability’ in philosophy. In [10] Fitch addresses the problematic question whether what is true can become known. It is considered problematic (paradoxical even) that the existence of unknown truths is inconsistent with the requirement that all truths are

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knowable. Again, the Moore-sentence $p \wedge \neg Kp$ provides the prime example: it cannot become known, because $K(p \wedge \neg Kp)$ entails an inconsistency under the standard interpretation of knowledge. For an overview of the literature on Fitch's paradox, see [7], we will later discuss some of that in detail, mainly Tennant's proposal on cartesian formulas [25]. The suggestion to interpret 'knowable' as 'known after an announcement' was made by van Benthem in [26].

Of course, some things *can* become known. For example, true facts p can always become known by announcing them, formalized as $p \rightarrow \langle p \rangle Kp$ ('if the atom p is true, then after announcing p, p is known')—the above-mentioned paradox involves announcement of epistemic information. One has to be careful with what one wishes for: some things can become known that were not true in the first place. Consider factual knowledge again: after announcing a fact, you also know that you know it. In other words 'knowledge of p' is knowable in the sense that there is an announcement that makes it true: we now have that $\langle p \rangle KKp$. But Kp was not true before that announcement, so this formula is not a knowable truth, except in the trivial sense when it was already true before the announcement.

Consider an extension of public announcement logic wherein we can express what becomes true, whether known or not, without explicit reference to announcements realizing that. Let us work our way upwards from a concrete announcement. When p is true, it becomes known by announcing it. Formally, in public announcement logic

$$\langle p \rangle Kp$$

which stands for 'the announcement of p can be made and after that the agent knows p'. More abstractly this means that there is a announcement ψ, namely $\psi = p$, that makes the agent know p, slightly more formal:

$$\text{there is a formula } \psi \text{ such that } \langle \psi \rangle Kp$$

We introduce a dynamic modal operator that expresses that:

$$\Diamond Kp$$

Obviously, the truth of this expression depends on the model: p has to be true. In case p is false, we can achieve $\Diamond K \neg p$ instead. The formula $\Diamond (Kp \vee K \neg p)$ is valid. Actually, we were slightly imprecise when suggesting that $\Diamond$ means 'there is a ψ such that'. In fact a restriction on ψ to purely epistemic formulas is required in the semantics, for a technical reason. The resulting logic is called arbitrary public announcement logic, $APAL$, or in short, *arbitrary announcement logic.*

Unlike the introductory examples so far, we present the logic as a *multi-agent* logic, wherein all knowledge operators are labelled with the knowing agent in question. For example, we write the validity above as $\Diamond (K_a p \vee K_a \neg p)$, indicating that this concerns what agent a can get to know. There are both conceptual and technical reasons for this multi-agent perspective. (i) Various paradoxical situations involving knowledge—that we can in principle also address in arbitrary announcement logic—require more than one agent (such as the Hangman Paradox, also known as the Surprise Examination, for a dynamic epistemic analysis see [29]). (ii) One technical reason is that: arbitrary announcement logic for more than one agent is strictly more expressive than public announcement logic, but that for a single agent it is equally expressive. (iii) We present interesting multi-agent formulations of knowability, such as knowledge transfer between agents and how to make distributive knowledge common knowledge.

Overview of contents In Section 2 we define the logical language $\mathcal{L}_{apal}$ and its semantics. This section also contains some technical tools repeatedly used in later sections. Section 3 shows various semantic results, including a 'knowable' fragment of the language (we do not fully characterize the knowable formulas), and an expressivity result: indeed our logic can express more than the public announcement logic upon which it is based. In Section 4 we provide a Hilbert-style axiomatization of arbitrary announcement logic. Section 5 discusses the generalization to a logic for arbitrary events.

2 Syntax and semantics

Both for the language and the structures we assume as background parameters a finite set of agents A and a countably infinite set of atoms P.

2.1 Syntactic notions

Definition 1 (Language) The language $\mathcal{L}_{apal}$ of arbitrary public announcement logic is inductively defined as

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

where $a \in A$ and $p \in P$. Additionally, $\mathcal{L}_{pal}$ is the language without inductive construct $\Box\varphi$, $\mathcal{L}_{el}$ the language without as well $[\varphi]\varphi$, and $\mathcal{L}_{pl}$ the language without as well $K_a\varphi$. The language with only $\Box$ as modal operator is $\mathcal{L}_{\Box}$.

The languages $\mathcal{L}_{pal}$, $\mathcal{L}_{el}$, and $\mathcal{L}_{pl}$ are those of *public announcement logic*, *epistemic logic*, and *propositional logic*, respectively. A formula in $\mathcal{L}_{el}$ is also called an *epistemic formula* and a formula in $\mathcal{L}_{pl}$ is also called a *propositional formula* or a *boolean*. For $K_a\varphi$, read 'agent a knows that φ'. For $[\varphi]\psi$, read '(if φ is true, then) after announcement of φ, ψ (is true)'. (Announcements are supposed to be public and truthful, and this is common knowledge among the agents.) For $\Box\psi$, read 'after every announcement, ψ is true'. Other propositional and epistemic connectives are defined by usual abbreviations. The dual of K_a is $\widehat{K}_a$, the dual of $[\varphi]$ is $\langle\varphi\rangle$, and the dual of $\Box$ is $\Diamond$. For $\widehat{K}_a\varphi$, read 'agent a considers it possible that φ', for $\langle\varphi\rangle\psi$, read '($\varphi$ is true and) after announcement of φ, ψ (is true)' and for $\Diamond\psi$, read 'there is an announcement after which ψ (is true).' Write P_φ for the set of atoms occurring in the formula φ (and similarly for necessity and possibility forms, below). Given some $P' \subseteq P$, $\mathcal{L}_x(P')$ is the logical language $\mathcal{L}_x$ ($\mathcal{L}_{apal}, \mathcal{L}_{el}, \dots$) restricted to atoms in P'.

Necessity and possibility forms A *necessity form* (see [14]) contains a unique occurrence of a special symbol $\sharp$. If $\boldsymbol{\psi}$ is such a necessity form (we write boldface Greek letters for arbitrary necessity forms) and $\varphi \in \mathcal{L}_{apal}$, then $\boldsymbol{\psi}(\varphi)$ is obtained from $\boldsymbol{\psi}$ by substituting φ for $\sharp$ in $\boldsymbol{\psi}$. Necessity forms are used to formulate the axiomatization of the logic, in Section 4, and in the proofs of several semantic results, in Section 3.

Definition 2 (Necessity forms) Let $\varphi \in \mathcal{L}_{apal}$. Then:

- $\sharp$ is a necessity form,
- if $\boldsymbol{\psi}$ is a necessity form then $(\varphi \rightarrow \boldsymbol{\psi})$ is a nec. form,

- if $\boldsymbol{\psi}$ is a necessity form then $[\varphi]\boldsymbol{\psi}$ is a nec. form,
- if $\boldsymbol{\psi}$ is a necessity form then $K_a\boldsymbol{\psi}$ is a nec. form.

We also use the dual notion of *possibility form*. It can be defined by the dual clauses to a necessity form: $\sharp$ is a possibility form, and if $\varphi \in \mathcal{L}_{apal}$ and $\boldsymbol{\psi}$ is a possibility form then $\varphi \wedge \boldsymbol{\psi}$, $\langle\varphi\rangle\boldsymbol{\psi}$, and $\hat{K}_a\boldsymbol{\psi}$ are possibility forms. To distinguish necessity forms from possibility forms we use different bracketing: write $\boldsymbol{\psi}\{\varphi\}$ for the possibility form with a unique occurrence of φ. For each necessity form $\boldsymbol{\psi}(\sharp)$ there is a possibility form $\boldsymbol{\psi'}\{\sharp\}$ such that for all φ, $\neg\boldsymbol{\psi}(\varphi)$ is logically equivalent to $\neg\boldsymbol{\psi'}\{\neg\varphi\}$.

2.2 Structural notions

Definition 3 (Structures) An *epistemic model* $M = (S, \sim, V)$ consists of a *domain* S of (factual) *states* (or 'worlds'), *accessibility* $\sim : A \to \mathcal{P}(S \times S)$, where each $\sim(a)$ is an equivalence relation, and a *valuation* $V : P \to \mathcal{P}(S)$. For $s \in S$, (M, s) is an *epistemic state* (also known as a pointed Kripke model). An *epistemic frame* S is a pair $(S, \sim)$. For a model we also write (S, V) and for a pointed model also (S, V, s).

For $\sim(a)$ we write $\sim_a$, and for $V(p)$ we write V_p; accessibility $\sim$ can be seen as a set of equivalence relations $\sim_a$, and V as a set of valuations V_p. Given two states s, s' in the domain, $s \sim_a s'$ means that s is indistinguishable from s' for agent a on the basis of its knowledge. We adopt the standard rules for omission of parentheses in formulas, and we also delete them in representations of structures such as (M, s) whenever convenient and unambiguous. Given a domain S of a model M, instead of $s \in S$ we also write $s \in M$.

Bisimulation Bisimulation is a well-known notion of structural similarity [6] that we will frequently use in examples and proofs, e.g. to achieve our expressivity results.

Definition 4 (Bisimulation) Let two models $M = (S, \sim, V)$ and $M' = (S', \sim', V')$ be given. A non-empty relation $\mathfrak{R} \subseteq S \times S'$ is a bisimulation between M and M' iff for all $s \in S$ and $s' \in S'$ with $(s, s') \in \mathfrak{R}$:

atoms for all $p \in P$: $s \in V_p$ iff $s' \in V'_p$;

forth for all $a \in A$ and all $t \in S$: if $s \sim_a t$, then there is a $t' \in S'$ such that $s' \sim'_a t'$ and $(t, t') \in \mathfrak{R}$;

back for all $a \in A$ and all $t' \in S'$: if $s' \sim'_a t'$, then there is a $t \in S$ such that $s \sim_a t$ and $(t, t') \in \mathfrak{R}$.

We write $(M, s) \leftrightarrows (M', s')$, iff there is a bisimulation between M and M' linking s and s', and we then call (M, s) and (M', s') bisimilar. The maximal bisimulation $\mathfrak{R}^{\mathsf{max}}$ between M and itself is an equivalence relation, and the result of identifying all $\mathfrak{R}^{\mathsf{max}}$ bisimilar worlds is a *minimal model* (also known as bisimulation contraction, or strongly extensional model) [1]. The construction preserves equivalence relations: if M is an epistemic model, its minimal model is also an epistemic model.

2.3 Semantics

Definition 5 (Semantics) Assume an epistemic model $M = (S, \sim, V)$. The interpretation of $\varphi \in \mathcal{L}_{apal}$ is defined by induction. Note the restriction to epistemic formulas in the clause for $\Box\varphi$.

$$\begin{array}{lll}
M, s \models p & \text{iff} & s \in V_p \\
M, s \models \neg\varphi & \text{iff} & M, s \not\models \varphi \\
M, s \models \varphi \wedge \psi & \text{iff} & M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models K_a\varphi & \text{iff} & \text{for all } t \in S : s \sim_a t \text{ implies } M, t \models \varphi \\
M, s \models [\varphi]\psi & \text{iff} & M, s \models \varphi \text{ implies } M|\varphi, s \models \psi \\
M, s \models \Box\varphi & \text{iff} & \text{for all } \psi \in \mathcal{L}_{el} : M, s \models [\psi]\varphi
\end{array}$$

In clause $[\varphi]\psi$ for public announcement, epistemic model $M|\varphi = (S', \sim', V')$ is defined as

$$\begin{array}{lll}
S' & = & \{s' \in S \mid M, s' \models \varphi\} \\
\sim'_a & = & \sim_a \cap (S' \times S') \\
V'_p & = & V_p \cap S'
\end{array}$$

Formula φ is valid in model M, notation $M \models \varphi$, iff for all $s \in S$: $M, s \models \varphi$. Formula φ is valid, notation $\models \varphi$, iff for all M: $M \models \varphi$.

The dynamic modal operator $[\varphi]$ is interpreted as an epistemic state transformer. Announcements are assumed to be truthful and public, and this is commonly known to all agents. Therefore, the model $M|\varphi$ is the model M restricted to all the states where φ is true, including access between states. Similarly, the dynamic model operator $\Box$ is interpreted as an epistemic state transformer. Note that in the definiendum of $\Box\varphi$ the announcements ψ in $[\psi]\varphi$ are restricted to purely epistemic formulas $\mathcal{L}_{el}$. This is motivated in depth, below. For the semantics of the dual operators, we have that $M, s \models \Diamond\psi$ iff there is a $\varphi \in \mathcal{L}_{el}$ such that $M, s \models \langle\varphi\rangle\psi$. In other words, $M, s \models \langle\varphi\rangle\psi$ iff $M, s \models \varphi$ and $M|\varphi, s \models \psi$. Write $[\![\varphi]\!]_M$ for the denotation of φ in M: $[\![\varphi]\!]_M := \{s \in S \mid M, s \models \varphi\}$. Given a sequence $\vec{\psi} = \psi_1 \dots \psi_k$ of announcements, write $M|\vec{\psi}$ for the model $(\dots(M|\psi_1)\dots|\psi_k)$ that is the result of the successive model restrictions.

The set of validities in our logic is called $APAL$. Formally this is relative to given sets of agents and atoms, but we will also use $APAL$ more informally to refer to arbitrary public announcement logic. Similarly for PL (propositional logic), EL (epistemic logic, a.k.a. $S5_n$ where $|A| = n$), and PAL (public announcement logic).

Bisimilar states satisfy the same epistemic formulas. This extends to $APAL$. The reader may easily verify that if the epistemic states (M, s) and (M', s') are bisimilar, then for all $\varphi \in \mathcal{L}_{apal}$: $(M, s) \models \varphi$ iff $(M', s') \models \varphi$.

Example 6 A valid formula of the logic is $\Diamond(K_a p \vee K_a\neg p)$. To prove this, let (M, s) be arbitrary. Either $M, s \models p$ or $M, s \models \neg p$. In the first case, $M, s \models \Diamond(K_a p \vee K_a\neg p)$ because $M, s \models \langle p\rangle(K_a p \vee K_a\neg p)$—the latter is true because ($M, s \models p$ and) $M|p, s \models K_a p \vee K_a\neg p$, because $M|p, s \models K_a p$. $M, s \models p$ and $M|p, s \models K_a p$; in the second case, we analogously derive $M, s \models \Diamond(K_a p \vee K_a\neg p)$ because $M, s \models \langle\neg p\rangle(K_a p \vee K_a\neg p)$.

This example also nicely illustrates the order in which arbitrary objects come to light. The meaning of $\models \Diamond\varphi$ is

for all (M, s) *there is an epistemic* ψ *such that* $M, s \models \langle\psi\rangle\varphi$ $\qquad (i)$.

This is really different from

$$\textit{there is an epistemic } \psi \textit{ such that for all } (M,s),\ M,s \models \langle\psi\rangle\varphi \qquad (ii),$$

which might on first sight be appealing to the reader, when extrapolating from the *incorrect* reading of $\models \Diamond\varphi$ as 'there is an epistemic ψ such that $\models \langle\psi\rangle\varphi$'. For example, there is no epistemic formula ψ such that $\langle\psi\rangle(K_a p \vee K_a \neg p)$ is valid. (Suppose there were. Then ψ would be valid; so an announcement of ψ would not be informative. Then, $\langle\psi\rangle(K_a p \vee K_a \neg p)$ would be equivalent to $K_a p \vee K_a \neg p$. But in any model where it is not known whether p the latter is false, so it is not valid. Contradiction.) In other words, (i) may be true, even when (ii) is false.

Motivation for the semantics of $\Box$ We now compare the given semantics for $\Box\varphi$ to two infelicitous alternatives, thus hoping to motivate our choice. The three options are (infelicitous alternatives are *-ed):

$$\begin{array}{rlll}
M,s \models \Box\varphi & \text{iff} & \text{for all } \psi \in \mathcal{L}_{el} : M,s \models [\psi]\varphi & \text{(Definition 5)} \\
*M,s \models \Box\varphi & \text{iff} & \text{for all } \psi \in \mathcal{L}_{apal} : M,s \models [\psi]\varphi & \text{(intuitive)} \\
*M,s \models \Box\varphi & \text{iff} & \text{for all } S' \subseteq S \text{ containing } s : M|S', s \models \varphi & \text{(structural)}
\end{array}$$

The 'intuitive' version for the semantics of $\Box\varphi$ more properly corresponds to its intended meaning 'φ is true after arbitrary announcements'. This would be a circular definition, as $\Box\varphi$ is itself one such announcement. It is not clear whether this is well-defined, but a restriction to announcements that are epistemic sentences seems at least reasonable in a context of knowledge and belief change.

The 'structural' version for the semantics of $\Box\varphi$ is more in accordance with one of Fine's proposals for quantification over propositional variables in modal logic [9]; his work strongly inspired our approach. This structural version is undesirable for our purposes, as it does not preserve bisimilarity of structures: two bisimilar states can now be separated because they may be in different subdomains. In dynamic epistemic logics it is considered preferable that action execution preserves bisimilarity; this is because bisimilarity implies logical equivalence, and we tend to think of such actions as changing the *theories* describing those structures, just as in belief revision. For an example, consider the following epistemic state $(M,1)$ – it consists of two states 1 and **1** where p is true and two states 0 and **0** where p is false; *linking* two states means that they are *indistinguishable* for the agent labeling the link; and the *underlined* state is the *actual* state.

$$\begin{array}{ccc}
\mathbf{0} & \text{---}\ a\ \text{---} & \mathbf{1} \\
| & & | \\
b & & b \\
| & & | \\
0 & \text{---}\ a\ \text{---} & \underline{1}
\end{array}$$

We have that $M,1 \models \Diamond(K_a p \wedge \neg K_b K_a p)$ for the structural $\Box$-semantics, as $M|\{1,\mathbf{1},\mathbf{0}\},1 \models K_a p \wedge \neg K_b K_a p$. On the other hand, for the $\Box$-semantics as defined, $M,1 \not\models \Diamond(K_a p \wedge \neg K_b K_a p)$, which can be easily seen as that formula is also false in the two-state structure $(M',1')$ depicted as

$$0' \text{ --- } a \text{ --- } \underline{1'}$$

where agent b can distinguish 0 from 1 but agent a cannot. Epistemic state $(M, 1)$ is bisimilar to $(M', 1')$, via the bisimulation $\mathfrak{R} = \{(0, 0'), (\mathbf{0}, 0'), (1, 1'), (\mathbf{1}, 1')\}$. We make two further observations concerning our preferred semantics '$\Box\varphi$ (is true) iff $[\psi]\varphi$ for all $\psi \in \mathcal{L}_{el}$'. First, given that truth is relative to a model, this semantics for $\Box$ amounts to '$\Box\varphi$ is true in (M, s) iff φ is true in all epistemistically definable submodels of M.' Second, note that public announcement logic is equally expressive as multiagent epistemic logic [23], so '$\Box\varphi$ (is true) iff $[\psi]\varphi$ for all $\psi \in \mathcal{L}_{el}$' corresponds to '$\Box\varphi$ (is true) iff $[\psi]\varphi$ for all $\psi \in \mathcal{L}_{pal}$.' So in fact we can replace boxes by announcements of any formula except those containing boxes—which comes fairly close to the intuitive interpretation again.

A theoretically quite justifiable and felicitous version of the 'structural' semantics for $\Box$ above would equate truth of $\Box\varphi$ with truth for all subsets of the minimal model (see page 4) of a model M, that contain the actual state s (in other words, a subset must not separate states that are in the maximal bisimulation relation on M). We did not explore this alternative semantics for $\Box$ in depth. For a given model there may be more such subsets than are epistemically definable, e.g., there may be uncountably many such subsets, whereas the epistemically definable subsets are countable.

3 Semantic results

3.1 Validities

3.1.1 Validities only involving $\Box$: S4

The following validities demonstrate the 'S4'-character of $\Box$. These validities do not, as usual, straightforwardly translate to frame properties, because we interpret $\Box$ as an epistemic state transformer and not by way of an accessibility relation. [1] It is also unclear if the set of validities only involving $\Box$ (i.e., $\mathcal{L}_\Box \cap APAL$) satisfies *uniform substitution* (replacing propositional variables by arbitrary formulas is validity preserving). See further research in Section 6.

Proposition 7 (S4 character of $\Box$) Let $\varphi, \psi \in \mathcal{L}_{apal}$. Then

1. $\models \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
2. $\models \Box\varphi \rightarrow \varphi$
3. $\models \Box\varphi \rightarrow \Box\Box\varphi$
4. $\models \varphi$ implies $\models \Box\varphi$

Proof

1. Obvious.

[1] It is possible to associate an accessibility relation to $\Box$. Given an model M, consider the union of its epistemically definable submodels, where we label copies of states (in order to distinguish them from their original) with an epistemic formula ψ representing (the class of formulas logically equivalent to ψ namely) $[\![\psi]\!]_M$. If $M|\varphi|\psi = M|\chi$, now add pair (s_φ, s_χ) to the accessibility relation R_ψ for announcement operator $[\psi]$. Let $R_\Box = \bigcup_{\psi \in \mathcal{L}_{el}} R_\psi$. If we do this just for announcements that correspond to sequences of announcements of a *single* epistemic formula ψ, the result is known as the *forest* for (M, s) and ψ [28].

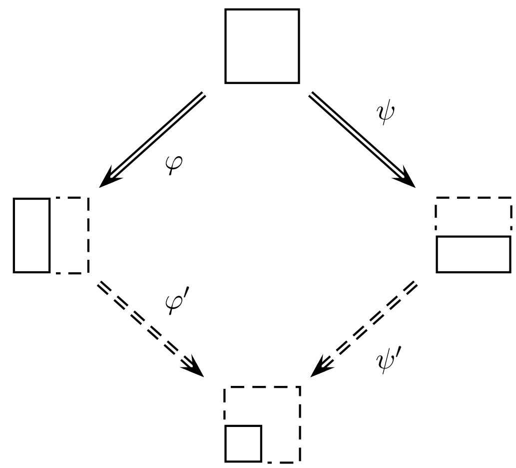


Figure 1: Church Rosser for announcements: given two announcements φ, ψ in some epistemic state (M, s), there are subsequent announcements φ', ψ' such that $(M|\varphi|\varphi', s)$ is bisimilar to $(M|\psi|\psi', s)$.

2. Assume $M, s \models \Box\varphi$. Then in particular, $M, s \models [\top]\varphi$, and therefore (as $M, s \models \top$) $M, s \models \varphi$.

3. Let M and $s \in M$ be arbitrary. Assume $M, s \models \Diamond\Diamond\neg\varphi$. Then there are epistemic χ and χ' such that $M, s \models \langle\chi\rangle\langle\chi'\rangle\neg\varphi$. Using the validity (for arbitrary formulas) $[\varphi][\varphi']\varphi'' \leftrightarrow [\varphi \wedge [\varphi]\varphi']\varphi''$, we therefore have $M, s \models \langle\chi \wedge [\chi]\chi'\rangle\neg\varphi$, from which follows $M, s \models \Diamond\neg\varphi$.

4. Let M, s be arbitrary. We have to show that for $\psi \in \mathcal{L}_{el}$: $M, s \models [\psi]\varphi$. From the assumption $\models \varphi$ follows $\models [\psi]\varphi$ by necessitation for $[\psi]$. Therefore also $M, s \models [\psi]\varphi$. As ψ is arbitrary, also $M, s \models \Box\varphi$.

$\Box$

3.1.2 Validities only involving $\Box$: MK and CR

Also valid are $\models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$ (McKinsey — MK) and $\models \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$ (Church-Rosser — CR). Axiom CR corresponds to the well-known frame property of confluence: $\forall xyz(Rxy \wedge Rxz \rightarrow \exists w(Ryw \wedge Rzw))$. In our terms, this can be formulated as follows. Given two distinct (and true) announcements φ, ψ in some epistemic state (M, s), then there are subsequent announcements φ', ψ' such that $(M|\varphi|\varphi', s)$ is bisimilar to $(M|\psi|\psi', s)$ (see Figure 1. The proofs of MK and CR are both somewhat involved and include lemmas and such—the first two lemmas take us to Proposition 10 showing validity of McKinsey and a subsequent trio of a lemma and two propositions takes us to Proposition 14 showing validity of Church-Rosser.

Lemma 8 Let $\varphi \in \mathcal{L}_{apal}$. Consider the set P_φ of atoms occurring in φ. Let M be a model where all states correspond on the valuation of P_φ. Then $M \models \varphi$ or $M \models \neg\varphi$, i.e., either φ or its negation is a model validity.

Proof Let $\varphi(\psi/p)$ be the substitution of ψ for all occurrences of p in formula φ. (Note the difference with the notation for necessity and possibility forms on page 3.) If p is true on M then $M \models \varphi \leftrightarrow \varphi(\top/p)$, otherwise $M \models \varphi \leftrightarrow \varphi(\bot/p)$. The result of successively substituting $\top$ or $\bot$ for all atoms in φ in that way is the formula $\varphi^\emptyset$. Clearly, $M \models \varphi \leftrightarrow \varphi^\emptyset$. As $\varphi^\emptyset$ does not contain atomic propositions, and given that $\models K_a\top \leftrightarrow \top$, $\models K_a\bot \leftrightarrow \bot$, $\models \Box\top \leftrightarrow \top$, and $\models \Box\bot \leftrightarrow \bot$, we have that $\models \varphi^\emptyset \leftrightarrow \top$ or $\models \varphi^\emptyset \leftrightarrow \bot$. Therefore, $M \models \varphi \leftrightarrow \top$ or $M \models \varphi \leftrightarrow \bot$, i.e., $M \models \varphi$ or $M \models \neg\varphi$. $\Box$

The characteristic formula δ^φ_s of the restriction of the valuation in a state s to the finite set P_φ of atoms occurring in φ, is defined as follows:

$$\delta^\varphi_s = \bigwedge\{p \mid p \in P_\varphi \text{ and } M, s \models p\} \wedge \bigwedge\{\neg p \mid p \in P_\varphi \text{ and } M, s \not\models p\}$$

Lemma 9 Let $\varphi \in \mathcal{L}_{apal}$ be arbitrary. Let M be a model, and s a world in M. Then $M|\delta^\varphi_s, s \models \varphi \rightarrow \Box\varphi$.

Proof As δ^φ_s is boolean we have that δ^φ_s is true in the model $M|\delta^\varphi_s$, i.e. $M|\delta^\varphi_s \models \delta^\varphi_s$, and remains true in any further restriction of M: for any formula $\psi \in \mathcal{L}_{el}$ we have that $M|\delta^\varphi_s|\psi \models \delta^\varphi_s$. As δ^φ_s is a conjunction of literals determining the values of all the atoms of φ, we have that for arbitrary epistemic formulas ψ, all states in models $M|\delta^\varphi_s|\psi$ correspond on the valuation of P_φ. By Lemma 8 we therefore have either $M|\delta^\varphi_s|\psi \models \varphi$ for any ψ, or $M|\delta^\varphi_s|\psi \models \neg\varphi$ for any ψ. In the former case $M|\delta^\varphi_s \models \Box\varphi$, and in the latter case $M|\delta^\varphi_s \models \Box\neg\varphi$. Hence $M|\delta^\varphi_s, s \models \varphi \rightarrow \Box\varphi$. $\Box$

Proposition 10 (MK is valid) $\models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

Proof Let M, s be arbitrary, and assume $M, s \models \Box\Diamond\varphi$. Consider the characteristic formula δ^φ_s of the valuation in s restricted to the atoms in φ. From $M, s \models \Box\Diamond\varphi$ and $M, s \models \delta^\varphi_s$ follows $M|\delta^\varphi_s, s \models \Diamond\varphi$. From that and twice Lemma 9 namely in (also valid) dual form $M|\delta^\varphi_s, s \models \Diamond\varphi \rightarrow \varphi$ and original form $M|\delta^\varphi_s, s \models \varphi \rightarrow \Box\varphi$ follows that $M|\delta^\varphi_s, s \models \Box\varphi$. Therefore $M, s \models \langle\delta^\varphi_s\rangle\Box\varphi$, thus $M, s \models \Diamond\Box\varphi$. $\Box$

We now proceed with matters towards proving Church-Rosser. We extend the substitution notation already in use ($\varphi(\psi/p)$ is the substitution of ψ for all occurrences of p in formula φ) to simultaneous substitution for infinite sequences $\varphi(\psi_0/p_0, \psi_1/p_1, \dots)$.

Lemma 11 Let $Q = \{q_n \mid n \in \mathbb{N}\} \subseteq P$ be an infinite set of atoms, let $\theta \in \mathcal{L}_{el}$ be an epistemic formula such that $P_\theta \cap Q = \emptyset$, and let $\varphi \in \mathcal{L}_{apal}$ with $P_\varphi \cap Q = \emptyset$. Given a frame S and a valuation V on S, there exists a valuation V' on S such that:

1. $[\![\varphi]\!]_{\mathsf{S},V'} = [\![\varphi]\!]_{\mathsf{S},V}$;
2. for all $\theta' \in \mathcal{L}_{el}$:

$$\begin{array}{rcl} [\![\theta']\!]_{\mathsf{S},V'} & = & [\![\theta'(\theta/q_0, q_0/q_1, \dots, q_n/q_{n+1}, \dots)]\!]_{\mathsf{S},V} \ ; \\ [\![\theta']\!]_{\mathsf{S},V} & = & [\![\theta'(q_1/q_0, \dots, q_{n+1}/q_n, \dots)]\!]_{\mathsf{S},V'} \end{array}$$

3. $[\![q_0]\!]_{\mathsf{S},V'} = [\![\theta]\!]_{\mathsf{S},V'} = [\![\theta]\!]_{\mathsf{S},V}$.

Proof The valuation V' needed is given by putting $V'(p) := V(p)$ for $p \notin Q$, $V'(q_0) := [\![\theta]\!]_{\mathsf{S},V}$, and for all $n \in \mathbb{N}$: $V'(q_{n+1}) := V(q_n)$. $\Box$

As a consequence of clause (2) of Lemma 11, we have that the epistemically definable subsets of (S, V) are the same as those of (S, V'). We now use the lemma to show that:

Proposition 12 If $M, s \models \Diamond\psi$ and $p \notin P_\psi$, then there exists a model M' only differing from M in the valuation of atoms not occurring in ψ such that $M', s \models \langle p \rangle \psi$.

Proof Let $M = (\mathsf{S}, V) = (S, \sim, V)$. We use first the above Lemma 11, namely for

$$Q := P \setminus P_\psi,\ q_0 := p,\ \theta = \top, \text{ and } \varphi := \Diamond\psi,$$

obtaining a new valuation V' s.t. $V'(p) = S$ and $[\![\Diamond\psi]\!]_{\mathsf{S},V'} = [\![\Diamond\psi]\!]_{\mathsf{S},V}$. Therefore, there must exist some $\theta \in \mathcal{L}_{el}$ such that $(\mathsf{S}, V', s) \models \langle\theta\rangle\psi$, so $s \in [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V'}$. Further, we can assume that $p \notin P_\theta$: the valuation of p has been set to $\top$ in V', therefore if there had been occurrences of p in θ they could have been replaced by $\top$. We now apply Lemma 11 again, with

$$Q := P \setminus (P_\theta \cup P_\psi),\ q_0 := p,\ \varphi := \langle\theta\rangle\psi, \text{ and } \theta \text{ as given,}$$

obtaining V'' such that $s \in [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V'} = [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V''}$ and $[\![p]\!]_{\mathsf{S},V''} = [\![\theta]\!]_{\mathsf{S},V''} = [\![\theta]\!]_{\mathsf{S},V'}$. Hence we obtain that: $s \in [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V'} = [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V''} = [\![\langle p\rangle\psi]\!]_{\mathsf{S},V''}$. $\Box$

Proposition 12 can be generalized to:

Proposition 13 Given a possibility form $\boldsymbol{\eta}$. If $M, s \models \boldsymbol{\eta}\{\Diamond\psi\}$ and $p \notin (P_\eta \cup P_\psi)$, then there exists a model M' only differing from M in the valuation of p such that $M', s \models \boldsymbol{\eta}\{\langle p\rangle\psi\}$.

Proof The proof is straightforward and by induction on the complexity of possibility forms. The basic case is the proof of Proposition 12. The case 'conjunction' starts with $M, s \models \chi \wedge \Diamond\psi$ and $p \notin P_\chi \cup P_\psi$. Etc. $\Box$

We will use Proposition 13 below, to prove the soundness of a derivation rule in the axiomatization of arbitrary announcement logic. For now, we only need Proposition 13 to show the CR property.

Proposition 14 (CR is valid) $\models \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$

Proof Suppose that CR fails. Then there exist M, s and φ such that $M, s \models \Diamond\Box\varphi \wedge \Diamond\Box\neg\varphi$. By applying Proposition 13 twice (namely for the possibility form 'conjunction', once for the left conjunct and once for the right conjunct) there are $p, q \notin P_\varphi$ and a model M' that is like M except for the valuation of p and q, such that $M', s \models \langle p\rangle\Box\varphi \wedge \langle q\rangle\Box\neg\varphi$. We therefore also have $M', s \models \langle p\rangle[q]\varphi \wedge \langle q\rangle[p]\neg\varphi$ from which follows $M', s \models \langle p\rangle\langle q\rangle\varphi \wedge \langle q\rangle\langle p\rangle\neg\varphi$, and therefore, as p and q are boolean (sequential announcement of booleans corresponds to the announcement of their conjunction), $M', s \models \langle p \wedge q\rangle(\varphi \wedge \neg\varphi)$, which is a contradiction. $\Box$

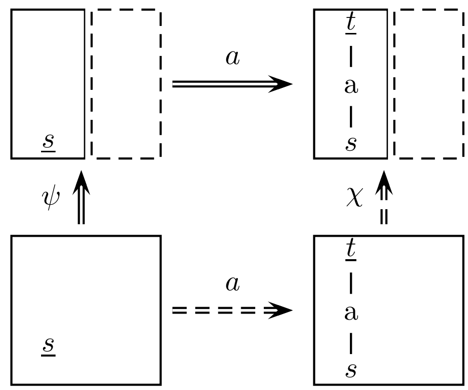


Figure 2: Illustration of the principle $K_a\Box\varphi \to \Box K_a\varphi$. Given $\langle\psi\rangle\hat{K}_a\neg\varphi$, there is a χ such that $\hat{K}_a\langle\chi\rangle\neg\varphi$.

3.1.3 The relation between knowledge and arbitrary announcement

Proposition 15 Let $\varphi \in \mathcal{L}_{apal}$. Then $\models K_a\Box\varphi \to \Box K_a\varphi$.

Proof Suppose $M, s \models K_a\Box\varphi$ and $M, s \models \psi$. Assume $t \in M|\psi$ with $s \sim_a t$. We have to prove that $M|\psi, t \models \varphi$. Because state t is also in M, from the assumption $M, s \models K_a\Box\varphi$ and (in M) $s \sim_a t$ follows $M, t \models \Box\varphi$. As ψ is true in t, $M|\psi, t \models \varphi$. $\Box$

Proposition 15 is visualized in Figure 2. Although $K_a\Box\varphi \to \Box K_a\varphi$ is valid, the other direction $\Box K_a\varphi \to K_a\Box\varphi$ is not valid. It is instructive to give a counterexample.

Example 16 ($\Box K_a\varphi \to K_a\Box\varphi$ is not valid) Consider the model:

$$\mathbf{0} \text{ — } b \text{ — } 1 \text{ — } a \text{ — } \underline{0}$$

We now have that $M, 0 \models \hat{K}_a\langle\hat{K}_b p\rangle(K_a p \wedge \neg K_b p)$, hence $M, 0 \models \hat{K}_a\Diamond(K_a p \wedge \neg K_b p)$. On the other hand $M, 0 \not\models \Diamond(\hat{K}_a(K_a p \wedge \neg K_b p)$, because $K_a p \wedge \neg K_b p$ is only true in the model restriction $\{\mathbf{0}, 1\}$ that *excludes* the actual state 0. Therefore, $\Box K_a\varphi \to K_a\Box\varphi$ is invalid. In simple words, it may unfortunately happen that we jump to a state where a model restriction is possible that excludes the actual state. Therefore, things that are true at that state may be impossible to realize by a reversal of that process.

3.1.4 Validities relating booleans and arbitrary announcements

The following Proposition 17 will be helpful to show that in the single-agent case every formula is equivalent to an epistemic $\mathcal{L}_{el}$-formula, as discussed in Subsection 3.2.

Proposition 17 Let $\varphi, \varphi_0, \ldots, \varphi_n \in \mathcal{L}_{pl}$ and $\psi \in \mathcal{L}_{apal}$.

1. $\models \Box\varphi \leftrightarrow \varphi$
2. $\models \Box\hat{K}_a\varphi \leftrightarrow \varphi$
3. $\models \Box K_a\varphi \leftrightarrow K_a\varphi$

4. $\models \Box(\varphi \vee \psi) \leftrightarrow (\varphi \vee \Box\psi)$

5. $\models \Box(\hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \cdots \vee K_a\varphi_n) \leftrightarrow (\varphi_0 \vee K_a(\varphi_0 \vee \varphi_1) \vee \ldots \vee K_a(\varphi_0 \vee \varphi_n))$

Proof In the proof, we use the dual (diamond) versions of all propositions.

1. $\models \Diamond\varphi \leftrightarrow \varphi$
   This is valid because $\langle\psi\rangle\varphi \leftrightarrow \varphi$ is valid in PAL, for any ψ and boolean φ.

2. $\models \Diamond K_a\varphi \leftrightarrow \varphi$
   Right-to-left holds because $\varphi \rightarrow \langle\varphi\rangle K_a\varphi$ is valid in PAL for booleans. The other way round, $\models \Diamond K_a\varphi \rightarrow \varphi$ because $\Diamond K_a\varphi \rightarrow \Diamond\varphi$ is valid in PAL, and $\Diamond\varphi \leftrightarrow \varphi$ is valid in PAL as we have seen above (φ being boolean).

3. $\models \Diamond\hat{K}_a\varphi \leftrightarrow \hat{K}_a\varphi$
   Right-to-left holds follows from the dual form of the validity $\Box\varphi \rightarrow \varphi$ (Proposition 7). Left-to-right holds because $\langle\psi\rangle\hat{K}_a\varphi \rightarrow \hat{K}_a\varphi$ is valid in PAL for booleans φ.

4. $\models \Diamond(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \Diamond\psi$
   Left-to-right: First, $\Diamond$ distributes over $\wedge$, and second, $\models \Diamond\varphi \leftrightarrow \varphi$ as we have established above. From right-to-left: $\varphi \wedge \Diamond\psi$ is equivalent to (apply case 1) $\Box\varphi \wedge \Diamond\psi$. From the semantics of $\Box$ now directly follows $\Diamond(\varphi \wedge \psi)$.

5. $\models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1 \wedge \ldots \wedge \hat{K}_a\varphi_n) \leftrightarrow \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1) \wedge \ldots \wedge \hat{K}_a(\varphi_0 \wedge \varphi_n)$
   We show this case for $n = 1$.

   Left-to-right: Directly in the semantics. Let M, s be arbitrary and suppose $M, s \models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1)$. Let ψ be the epistemic formula such that $M, s \models \langle\psi\rangle(K_a\varphi_0 \wedge \hat{K}_a\varphi_1)$. In the model $M|\psi$ we now have that $M|\psi, s \models K_a\varphi_0$ so $M|\psi, s \models \varphi_0$. Also $M|\psi, s \models \hat{K}_a\varphi_1$. Let t be such that $s \sim_a t$ and $M|\psi, t \models \varphi_1$. As $M|\psi, s \models K_a\varphi_0$, and $s \sim_a t$, also $M|\psi, t \models \varphi_0$. Therefore $M|\psi, t \models \varphi_0 \wedge \varphi_1$, and therefore $M|\psi, s \models \hat{K}_a(\varphi_0 \wedge \varphi_1)$. So $M|\psi, s \models \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1)$, and as φ_0 and φ_1 are booleans also $M, s \models \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1)$.[2]

   Right-to-left: For the other direction, suppose $M, s \models \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1)$. Consider the model $M|\varphi_0$. Because $M, s \models \hat{K}_a(\varphi_0 \wedge \varphi_1)$, and φ_1 is boolean, there must be a $t \in M|\varphi_0$ such that $M|\varphi_0, t \models \varphi_1$. So $M|\varphi_0, s \models \hat{K}_a\varphi_1$. Also $M|\varphi_0, s \models K_a\varphi_0$, because φ_0 is boolean. So $M|\varphi_0, s \models K_a\varphi_0 \wedge \hat{K}_a\varphi_1$ and therefore $M, s \models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1)$.

□

3.2 Expressivity

If there is a single agent only, arbitrary announcement logic reduces to epistemic logic. But for more than one agent, it is strictly more expressive than public announcement logic. We remind the reader that, in the absence of common knowledge, public announcement logic is equally expressive as epistemic logic.

First, we consider the single-agent case. Let $A = \{a\}$. We obtain the result by applying Proposition 17. We need some additional terminology as well. A formula is in *normal form* when it is a conjunction of disjunctions of the form $\varphi \vee \hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \ldots \vee K_a\varphi_n$, where

[2]Alternatively, one can use more straightforwardly the $S5$ validity $(K_a\varphi_0 \wedge \hat{K}_a\varphi_1) \rightarrow (K_a\varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1))$

$\varphi, \varphi_0, \ldots, \varphi_n$ are all formulas in propositional logic. Every formula in single-agent *S5* is equivalent to a formula in normal form [18]. A normal form may not exist for a multi-agent formula, e.g., it does not exist for $K_a K_b p$. This explains why the result below does not carry over to the multi-agent case.

Proposition 18 Single agent arbitrary announcement logic is equally expressive as epistemic logic.

Proof We prove by induction on the number of occurrences of $\Box$, that every formula in single-agent arbitrary announcement logic is equivalent to a formula in epistemic logic. Put the epistemic formula in the scope of an innermost $\Box$ in normal form. First, we distribute $\Box$ over the conjunction (Proposition 7.1). We now get formulas of the form $\Box(\varphi \vee \hat{K}_a \varphi_0 \vee K_a \varphi_1 \vee \cdots \vee K_a \varphi_n)$. These are reduced by application of Propositions 17.4 and 17.5 to formulas $(\varphi \vee \varphi_0) \vee K_a(\varphi_0 \vee \varphi_1) \vee \cdots \vee K_a(\varphi_0 \vee \varphi_n)$. $\Box$

Proposition 19 Arbitrary announcement logic is strictly more expressive than epistemic logic.

Proof The proof follows an abstract argument. Suppose the logics are equally expressive, in other words, that there is some reduction rule for arbitrary announcement such that any formula can be reduced to an expression without $\Box$. Given the reduction of *PAL* to *EL*, this entails that every arbitrary announcement formula should be equivalent to an epistemic logical formula. Now the crucial observation is that this epistemic formula only contains a *finite* number of atomic propositions. We then construct models that cannot be distinguished in the restricted language, but can be distinguished in a language with more atoms.

So it remains to give a specific formula and a specific pair of models. Note that the formula must involve more than one agent, as single-agent arbitrary announcement logic is reducible to epistemic logic (see Proposition 18).

Consider the formula $\Diamond(K_a p \wedge \neg K_b K_a p)$. Assume, towards a contradiction, that it is equivalent to an epistemic logical formula ψ. W.l.o.g. we may assume that ψ only contains the atom p.[3] We now construct two different epistemic states (M, s) and (M', s') involving a *new* atom q such that $\Diamond(K_a p \wedge \neg K_b K_a p)$ is false in the first but true in the second. We also take care that the two models are bisimilar with respect to the language without q. Therefore, the supposed reduction is either true in both models or false in both models. Contradiction. Therefore, no such reduction exists.

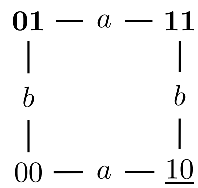
The required models are as follows. Epistemic state $(M, 1)$ consists of the well-known model M where a cannot distinguish between states where p is true and false, but b can (but knows that a cannot, etc.), i.e., domain $\{0, 1\}$ with universal access for a and identity access for b, where p is only true at 1, and 1 is the actual state. Visualized as:

$$0 \text{ --- } a \text{ --- } \underline{1}$$

Epistemic state $(M', 10)$ consists of two copies of M, namely one where a new fact q is true

[3] The alternative is that ψ contains a *finite* number of atoms. What other atoms apart from p? It does not matter: the contradiction on which the proof of Proposition 19 is based, merely requires a 'fresh' atom not yet occurring in ψ.

and another one where q is false. In the actual state 10, q is false. We visualize this as:



We now have that $(M, 1)$ is bisimilar to $(M', 10)$ with regard to the epistemic language for atom p and agents a, b. Therefore $M, 1 \models \psi$ iff $M', 10 \models \psi$. On the other hand $(M, 1)$ is not bisimilar to $(M', 10)$ with regard to the epistemic language for atoms p, q and agents a, b. This is evidenced by the fact that $M, 1 \not\models \Diamond(K_a p \land \neg K_b K_a p)$ but, instead, $M', 10 \models \Diamond(K_a p \land \neg K_b K_a p)$. The latter is because $M', 10 \models \langle p \lor q \rangle (K_a p \land \neg K_b K_a p)$: the announcement $p \lor q$ restricts the domain to the three states where it is true, and $M'|(p \lor q), 10 \models K_a p \land \neg K_b K_a p$, because $10 \sim_b 11$ and $M'|(p \lor q), 11 \models \neg K_a p$.[4] $\Box$

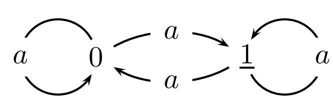
As an aside, because it departs from our assumption that all accessibility relations are equivalence relations, we have yet another result concerning expressive power. Consider the more general *multi-agent models* $M = (S, R, V)$ for accessibility functions $R : A \to \mathcal{P}(S \times S)$. Unlike the corresponding relations $\sim_a$ in epistemic models, the relations R_a are not necessarily equivalence relations. We now interpret the same language on those structures, with the obvious (only) difference that $M, s \models K_a \varphi$ iff for all $t \in S : R_a(s, t)$ implies $M, t \models \varphi$. Many results still carry over to the more general logic, but the expressivity results are now different.

Proposition 20 With respect to the class of multi-agent models for (a single) accessibility relation R_a, single-agent arbitrary announcement logic is strictly more expressive than public announcement logic.

Proof Along the same argument as in Proposition 19, on the assumption that a given formula φ is logically equivalent to a $\Box$-free formula ψ not containing some fresh atom q we present two models that are bisimilar with respect to the atoms in ψ and that therefore cannot be distinguished by ψ, but that have a different valuation for φ. From the contradiction follows strictly larger expressivity.

Consider the formula $\Diamond(K_a p \land \neg K_a K_a p)$ and assume that it is equivalent to an epistemic ψ only containing atom p; and consider models M and M' as follows:

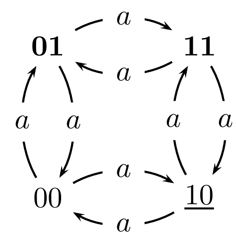
Multi-agent state $(M, 1)$ consists of the familiar model M where a cannot distinguish between states 1 and 0 where p is true and false, respectively, and where 1 is the actual state. We now explicitly visualize all pairs in the accessibility relation and get:



Multi-agent state $(M', 10)$ consists of two copies of M, namely a bottom one where a new fact q is false and a top one where q is true. The actual state is 10. Accessibility relations

[4] Kooi, in a personal communication, suggested an interesting alternative proof of larger expressivity that does not require a fresh atom, but deeper and deeper modal nesting. The proof is almost the same as the one we present here, but rather than an atom that distinguishes the worlds there are strings of worlds of different length attached to each world of the square.

are as shown—note that there is no reflexive access on any world.



We now have that $(M, 1)$ is bisimilar to $(M', 10)$ with regard to the epistemic language for atom p and agent a, but that $(M, 1)$ is not bisimilar to $(M', 10)$ with regard to the epistemic language for atoms p, q and agent a. Therefore, $M, 1 \models \psi$ iff $M', 10 \models \psi$. On the other hand, $M, 1 \not\models \Diamond(K_a p \wedge \neg K_a K_a p)$ but $M', 10 \models \Diamond(K_a p \wedge \neg K_a K_a p)$, as $M', 10 \models \langle p \vee q \rangle (K_a p \wedge \neg K_a K_a p)$. □

3.3 Compactness and model checking

Compactness The counterexample used in the proof of Proposition 19 can be adjusted to show that $APAL$ is not compact.

Proposition 21 Arbitrary announcement logic is not compact.

Proof Take the following infinite set of formulas:

$$\{[\theta](K_a p \rightarrow K_b K_a p) \mid \theta \in \mathcal{L}_{el}\} \cup \{\neg\Box(K_a p \rightarrow K_b K_a p)\}.$$

By the semantics of $\Box$, this set is obviously not satisfiable. But we will show that *any of its finite subsets is satisfiable*. This contradicts compactness. Let

$$\{[\theta_i](K_a p \rightarrow K_b K_a p) \mid 0 \leq i \leq n\} \cup \{\neg\Box(K_a p \rightarrow K_b K_a p)\}$$

be any such finite subset, and let q be an atomic sentence that is distinct from p and does not occur in any of the sentences θ_i $(0 \leq i \leq n)$. Take now the epistemic state $(M', 10)$ as in the proof of Proposition 19. As shown above, we have $M', 10 \models \Diamond(K_a p \wedge \neg K_b K_a p)$, and thus $M', 10 \models \neg\Box(K_a p \rightarrow K_b K_a p)$. On the other hand, for the epistemic state $(M, 1)$ as in the above proof, we have shown above that we have $M, 1 \not\models \Diamond(K_a p \wedge \neg K_b K_a p)$, i.e. $M, 1 \models \Box(K_a p \rightarrow K_b K_a p)$. By the semantics of $\Box$, it follows that $M, 1 \models [\theta_i](K_a p \rightarrow K_b K_a p)$ for all $0 \leq i \leq n$; but q doesn't occur in any of these formulas, so their truth-values must be the same at $(M', 10)$ and $(M, 1)$ (since as shown above, the two epistemic states are bisimilar w.r.t. the language without q). Thus, we have $M', 10 \models [\theta_i](K_a p \rightarrow K_b K_a p)$ for all $0 \leq i \leq n$. Putting these together, we see that our finite set of formulas is satisfied at the state $(M', 10)$. □

Model checking We preferred to keep some technical results on model checking out of the paper. The model checking problem for the logic $APAL$ (to determine the extension of a given formula in a given model) is PSPACE-complete (Work in progress by Balbiani et al.).

Let us briefly sketch why the model checking problem for $APAL$ is decidable. This result is not trivial, because of the implicit quantification over *all* atoms in the $\Box$-operator. Consider

positive	$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\neg\varphi]\varphi \mid \Box\varphi$
preserved	$\models \varphi \rightarrow \Box\varphi$
successful	$\models [\varphi]\varphi$
knowable	$\models \varphi \rightarrow \Diamond K_a\varphi$

Table 1: Overview of formula properties. A formula satisfying the condition in the right column is said to have the property in the left column.

a finite model with a recursive valuation map (from the infinite set of atomic sentences to the powerset of the model). It is well-known that determining the largest bisimulation on such a model is a decidable problem, and so is finding all subsets of the model that are closed under the largest bisimulation. Given such a model and a formula, we can then then replace all occurrences of $\Box\varphi$ in that formula by a finite conjunction of announcement sentences $[\theta]\varphi$, where the denotation of the announced formulas θ ranges over all the subsets that are closed under the largest bisimulation of the model. (We use here the known fact that a subset of a finite model is definable in basic modal/epistemic logic if and only if it is closed under the largest bisimulation.) To determine the truth of the resulting formula one can then use a model-checking algorithm for public announcement logic.

Decidability The issue of the decidability of the logic has been resolved by French and van Ditmarsch in [11]: arbitrary announcement logic is *undecidable*. A logic is decidable iff there is a terminating procedure to determine whether a given formula is satisfiable. French and van Ditmarsch proved via a tiling argument (and an embedding) that it is co-RE complete to determine whether a given formula can be satisfied in some model.

3.4 Knowability and other semantic or syntactic fragments

A suitable direction of research is the syntactic or semantic characterization of interesting fragments of the logic. In this section we define *positive*, *preserved*, *successful*, and *knowable* formulas, and investigate their relation. (See Figure 1 for an overview of definitions.)

The *positive formulas* intuitively correspond to formulas that do not express ignorance; in epistemic logical ($\mathcal{L}_{el}$) terms: in which negations do not precede K_a operators. We consider a generalization of that notion to $\mathcal{L}_{apal}$. The fragment of the *positive formulas* is inductively defined as

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\neg\varphi]\varphi \mid \Box\varphi$$

Note that the truth of the announcement is a *condition* of its execution, which, when seen as a disjunction, explains the negation in $[\neg\varphi]$. Unfortunately, the negation in $[\neg\varphi]\varphi$ makes 'positive' somewhat of a misnomer.

The *preserved formulas* preserve truth under arbitrary (epistemically definable) model restriction, also known as relativization. They are (semantically) defined as those φ for which $\models \varphi \rightarrow \Box\varphi$.[5] There is no corresponding semantic principle in public announcement logic that expresses truth preservation.

We now prove that positive formulas are preserved. Restricted to epistemic logic without common knowledge, this was observed by van Benthem in [27]. In [29] van Ditmarsch and

[5] In [21], the same semantic condition defines the *persistent* formulas.

Kooi extended van Benthem's result, with an additional clause $[\neg\varphi]\varphi$. (And also, unlike here, an additional clause $C_B\varphi$ for subgroup common knowledge operators, where $B \subseteq A$.) Surprisingly, we can further extend the notion of 'positive' to arbitrary announcement logic, by adding a clause $\Box\varphi$: in the case $\Box\varphi$ of the inductive proof below to show truth preservation, assuming the opposite easily leads to a contradiction.

Proposition 22 Positive formulas are preserved.

Proof For "M' *is a submodel of* M" write $M' \subseteq M$. To prove the proposition it is sufficient to show the following:

> Given M, M' with $M' \subseteq M$, a state s in the domain of M', and a positive formula φ. If $(M, s) \models \varphi$, then $(M', s) \models \varphi$ (i).

It is sufficient, because it then also holds for all *epistemically definable* submodels M'. We show (i) by proving an even slightly stronger proposition, namely:

> Given M, M', M'' with $M'' \subseteq M' \subseteq M$, state s in the domain of M'', and positive φ. If $(M', s) \models \varphi$, then $(M'', s) \models \varphi$.

This has the advantage of loading the induction hypothesis. Loading is needed for the case $[\neg\varphi]\psi$ of the proof, that is by induction on the formula. We assume most cases to be well-known, except for the case $[\neg\varphi]\psi$, similarly shown in [29], and $\Box\varphi$, which is new.

Case $[\neg\varphi]\psi$:
Given is $(M', s) \models [\neg\varphi]\psi$. We have to prove that $(M'', s) \models [\neg\varphi]\psi$. Assume that $(M'', s) \models \neg\varphi$. By using the contrapositive of the induction hypothesis, $(M', s) \models \neg\varphi$. From that and the assumption $(M', s) \models [\neg\varphi]\psi$ follows $(M'|\neg\varphi, s) \models \psi$. Because $(M', s) \models \neg\varphi$, $M''|\neg\varphi$ is a submodel of $M'|\neg\varphi$. From $(M'|\neg\varphi, s) \models \psi$ and $M''|\neg\varphi \subseteq M'|\neg\varphi \subseteq M' \subseteq M$ it follows from (the loaded version of!) induction that $(M''|\neg\varphi, s) \models \psi$. Therefore $(M'', s) \models [\neg\varphi]\psi$.

Case $\Box\varphi$:
Assume $(M', s) \models \Box\varphi$. Suppose towards a contradiction that $(M'', s) \not\models \Box\varphi$. Then there is a ψ such that $(M'', s) \models \langle\psi\rangle\neg\varphi$, from which follows $(M''|\psi, s) \not\models \varphi$. From $M''|\psi \subseteq M'' \subseteq M'$ and contraposition of induction follows $(M', s) \not\models \varphi$. But from $(M', s) \models \Box\varphi$ follows $(M', s) \models [\top]\varphi$ which equals $(M', s) \models \varphi$ that contradicts the previous. □

Van Benthem [27] also shows that preserved formulas are (logically equivalent to) positive. This is not known for the extension of these notions to public announcement logic in [29], nor for arbitrary announcement logic. An answer to this question seems hard.

Another semantic notion is that of *success*. *Successful formulas* are believed after their announcement, or, in other words, after 'revision' with that formula. This corresponds to the postulate of 'success' in AGM belief revision. Formally, φ is a *successful formula* iff $[\varphi]\varphi$ is valid (see [29], elaborating an original but slightly different proposal in [12]). The validity of $[\varphi]\varphi$ is equivalent to the validity of $\varphi \to [\varphi]K_a\varphi$: "if φ is true, then after announcing φ, φ is believed." (see [29]). This validity describes in a dynamic epistemic setting the postulate of success for belief expansion: "if φ is true, then after expansion with φ, φ should be believed."

Proposition 23 Preserved formulas are successful.

Proof $\models \varphi \to \Box\varphi$ implies $\models \varphi \to [\varphi]\varphi$, and $\models \varphi \to [\varphi]\varphi$ iff $\models [\varphi]\varphi$. □

Corollary 24 Positive formulas are successful.

Fitch observed that not all unknown truths can become known [10, 7], such as the well-known $p \wedge \neg Kp$. Instead of calling this a paradox (which Fitch did not do either!), we prefer to call it a fact, and the question then is what unknown truths *can* become known. For a single agent a we can define the *knowable formulas* as those for which $\models \varphi \rightarrow \Diamond K_a\varphi$, and the most obvious multi-agent version defines the *knowable formulas* as those for which, for *all* agents $a \in A$, $\models \varphi \rightarrow \Diamond K_a\varphi$. (See a paragraph below for some additional multi-agent versions of knowability.) We can now observe that:

Proposition 25 Positive, preserved, and successful formulas are all knowable.

Proof Similar to the proof of Prop. 23. Observe that $\models \varphi \rightarrow \Box\varphi$ implies $\models \varphi \rightarrow [\varphi]\varphi$ which is equivalent to $\models \varphi \rightarrow [\varphi]K_a\varphi$; $\models \varphi \rightarrow [\varphi]K_a\varphi$ is equivalent to $\models \varphi \rightarrow \langle\varphi\rangle K_a\varphi$; and $\models \varphi \rightarrow \langle\varphi\rangle K_a\varphi$ implies $\models \varphi \rightarrow \Diamond K_a\varphi$. □

The syntactic characterization of knowable formulas remains an open question—but we would like to emphasize that, given a choice of interpretation for $\Box$ as in our logic, this has become a purely technical question. We think that this is a proper way to address knowability issues. Some knowable formulas are not positive, for example $\neg K_a p$: if true, announce $\top$, and $K_a\neg K_a p$ (still!) holds. Therefore $\models \neg K_a p \rightarrow \Diamond K_a \neg K_a p$.

Other approaches The excellent entry in the Stanford Encyclopedia of Philosophy on Fitch's Paradox [7] gives an overview of semantic and syntactic restrictions intended to avoid its paradoxical character.

It is relevant to mention Tennant's cartesian formulas: a formula φ is *cartesian* iff $K\varphi$ is not provably inconsistent [25]. A semantic correspondent of that, more in line with semantic features of formulas that we distinguished above, would be to define φ as cartesian iff $K\varphi$ is satisfiable, or, in other terms, iff $\not\models \neg K\varphi$. Van Benthem observed that cartesian formulas may not be knowable [26]. For example, the formula $p \wedge \neg Kq$ is cartesian but not knowable:

Consider a model where the formula is satisfied in a state wherein p is true but q is false. Now announce p. This results in a state where p is now known but $\neg Kq$ is of course still true. So, with introspection for knowledge and distribution of K over $\wedge$ we have that $K(p \wedge \neg Kq)$ is true. Therefore, the formula is cartesian.

On the other hand, we have that $\not\models p \wedge \neg Kq$, because in a model where the denotations of atoms p and q are the *same*, $p \wedge \neg Kq$ is false in any model restriction. Therefore, the formula is not knowable (in our sense).

It seems reasonable that this formula should be knowable in some other sense. But it is unclear in what sense. For example, what if one characterizes the knowable formulas as those for which for all agents—returning to the multiagent situation—$\varphi \rightarrow \Diamond K\varphi$ is merely *satisfiable*, and not necessarily valid? Unfortunately, *every* formula is knowable in that sense. If φ is valid, then $\Box K\varphi$ is valid, and $\varphi \rightarrow \Box K\varphi$ as well, so also $\varphi \rightarrow \Diamond K\varphi$, so a fortiori it is satisfiable. If φ is not valid, there must be an epistemic model M and a state s in that model where φ is false. But in that case we also have, trivially, that $M, s \models \varphi \rightarrow \Diamond K\varphi$. Therefore, $\varphi \rightarrow \Diamond K\varphi$ is satisfiable. Therefore, $\varphi \rightarrow \Diamond K\varphi$ is satisfiable for all φ.

Another (rather summary) syntactic characterization, within an intuitionistic setting, is that by Dummett in [8].

Moss and Parikh's topologic [21, 22] has the same language combining the knowledge operator K with box $\Box$, although for a single agent only. They interpret $\Box$ not in our temporal sense but in a spatial sense. With us, $\Diamond\varphi$ means 'φ is true after a sequence of announcements', i.e., 'after some time'. Moss and Parikh suggest to interpret $\Diamond\varphi$ as 'φ is true when taking some effort narrowing down the possibilities', i.e., 'closer'. How they relate K and $\Box$ in their semantics is different from our approach, because the structure on which they interpret their language is a topology of subsets of the domain of states. Most interestingly an open set in a topology is characterized by a 'knowability-like' formula: $M \models \varphi \to \Diamond K\varphi$ iff $[\![\varphi]\!]_M$ is an *open set*. (See [21, p.98]). An open set is a subset of the domain of the model M with a certain property relative to the topology defined on that domain.) They do not observe the relevance of their logic for knowability issues. Incidentally, Fitch leaves the question of how to interpret $\Box$ open in [10], and explicitly says that it does not have to be interpreted temporally: "the element of time will be ignored in dealing with these various concepts [such as knowledge]" [10, p.135].

Multi-agent versions of knowability There are various multi-agent versions of knowability that can be explored. To name a few:

- $\varphi \to \Diamond C_A\varphi$: commonly knowable truths
- $K_a\varphi \to \Diamond C_A\varphi$: an individual can *publish* his knowledge
- $K_a\varphi \to \Diamond K_b\varphi$: *knowledge transfer* from a to b
- $D_A\varphi \to \Diamond C_A\varphi$: distributed knowledge can be made common

Such notions are useful for the specification of both static and dynamic aspects of multi-agent systems, including properties of communication protocols. They suffer from similar constraints as the original Fitch knowability. For example, my knowledge that p is true and that you are ignorant of p, formalized as $K_{me}(p \land \neg K_{you}p)$, is not transferable to you, as $K_{you}(p \land \neg K_{you}p)$ is inconsistent for knowledge. The question what distributed knowledge can be made common, is relevant to compute the global consequences of local propagation of information through distributed networks.

4 Axiomatization

4.1 The axiomatization APAL and its soundness

We now provide a complete axiomatization of $\mathcal{L}_{apal}$.

Definition 26 The axiomatization **APAL** is given in Table 2. A formula is a *theorem* if it belongs to the least set of formulas containing all axioms and closed under the rules. If φ is a theorem, we write $\vdash \varphi$.

Proposition 27 (Soundness) The axiomatization **APAL** is sound. We only pay attention to the axiom and the derivation rule involving $\Box$.

1. $\Box\varphi \to [\psi]\varphi$, where $\psi \in \mathcal{L}_{el}$ (arbitrary and specific announcement)

all instantiations of propositional tautologies	
$K_a(\varphi \rightarrow \psi) \rightarrow (K_a\varphi \rightarrow K_a\psi)$	distribution of knowledge over implication
$K_a\varphi \rightarrow \varphi$	truth
$K_a\varphi \rightarrow K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \rightarrow K_a\neg K_a\varphi$	negative introspection
$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$	atomic permanence
$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$	announcement and negation
$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	announcement and conjunction
$[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$	announcement and knowledge
$[\varphi][\psi]\chi \leftrightarrow [(\varphi \wedge [\varphi]\psi)]\chi$	announcement composition
$\Box\varphi \rightarrow [\psi]\varphi$ where $\psi \in \mathcal{L}_{el}$	arbitrary and specific announcement
From φ and $\varphi \rightarrow \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of knowledge
From φ, infer $[\psi]\varphi$	necessitation of announcement
From $\psi \rightarrow [\theta][p]\varphi$, infer $\psi \rightarrow [\theta]\Box\varphi$ where $p \notin P_\psi \cup P_\theta \cup P_\varphi$	deriving arbitrary announcement / $R(\Box)$

Table 2: The axiomatization **APAL**

2. From $\psi \rightarrow [\theta][p]\varphi$, infer $\psi \rightarrow [\theta]\Box\varphi$, where $p \notin (P_\psi \cup P_\theta \cup P_\varphi)$ (deriving arbitrary announcement)

Proof

1. The soundness of 'arbitrary and specific announcement' follows directly from the semantics of $\Box$. The restriction to epistemic formulas is important. Without that restriction it is unclear if the axiom is sound.

2. To show the soundness of 'deriving arbitrary announcement', we first observe that the formulas $\psi \rightarrow [\theta][p]\varphi$ and $\psi \rightarrow [\theta]\Box\varphi$ are necessity forms, such that their negations are equivalent to possibility forms (see Definition 2 on page 3). We then use Proposition 13, which says that diamonds in possibility forms can be witnessed by fresh atoms.

   Suppose, towards a contradiction, that $\psi \rightarrow [\theta][p]\varphi$ is valid but that $\psi \rightarrow [\theta]\Box\varphi$ is *not* valid, i.e. we have a model such that $(\mathsf{S}, V, s) \models \neg(\psi \rightarrow [\theta]\Box\varphi)$. As it is the negation of a necessity form, formula $\neg(\psi \rightarrow [\theta]\Box\varphi)$ is equivalent to a possibility form $\boldsymbol{\chi}\{\Diamond\neg\varphi\}$. (Note that $\neg(\psi \rightarrow [\theta][p]\varphi)$ is therefore equivalent to the possibility form $\boldsymbol{\chi}\{\langle p\rangle\neg\varphi\}$.) From $(\mathsf{S}, V, s) \models \boldsymbol{\chi}\{\Diamond\neg\varphi\}$ and Proposition 13 follows that there exists a valuation V' and an atom $p \notin (P_\psi \cup P_\theta \cup P_\varphi)$ such that $(\mathsf{S}, V', s) \models \boldsymbol{\chi}\{\langle p\rangle\neg\varphi\}$. As fresh atom p we may choose the p in $\psi \rightarrow [\theta][p]\varphi$. So $(\mathsf{S}, V', s) \models \neg(\psi \rightarrow [\theta][p]\varphi)$. This contradicts the validity of $\psi \rightarrow [\theta][p]\varphi$.

$\Box$

4.2 Example derivations

Example 28 We show that the validity $\Box p \rightarrow \Box\Box p$ is also a theorem. In step 4 we use that the axiomatization for public announcement logic **PAL** satisfies the property of 'substitution of equivalents' (see [23, 24], or [31] for details). In step 8 of the derivation we use that $\Box p \rightarrow [q]\sharp$ is a necessity form, and in step 9 of the derivation we use that $\Box p \rightarrow \sharp$ is a necessity form.

1. $\vdash \Box p \rightarrow [q \wedge (q \rightarrow r)]p$ — arbitrary and specific announcement
2. $\vdash (q \rightarrow r) \leftrightarrow [q]r$ — atomic permanence
3. $\vdash (q \wedge (q \rightarrow r)) \leftrightarrow (q \wedge [q]r)$ — 2, propositionally
4. $\vdash [q \wedge (q \rightarrow r)]p \leftrightarrow [q \wedge [q]r]p$ — substitution of equivalents for **PAL** (*)
5. $\vdash [q \wedge [q]r]p \leftrightarrow [q][r]p$ — announcement composition
6. $\vdash [q \wedge (q \rightarrow r)]p \leftrightarrow [q][r]p$ — 4, 5, propositionally
7. $\vdash \Box p \rightarrow [q][r]p$ — 1, 6, propositionally
8. $\vdash \Box p \rightarrow [q]\Box p$ — 7, deriving arbitrary announcement
9. $\vdash \Box p \rightarrow \Box\Box p$ — 8, deriving arbitrary announcement

Example 29 For another example, we show that $[\Box p]p$ is a theorem. This means that, regardless of the restriction in axiom $\Box\varphi \rightarrow [\psi]\varphi$ (arbitrary and specific announcement) that $\psi \in \mathcal{L}_{el}$, there are already very basic theorems of the form $[\psi]\varphi$ where ψ is *not* an epistemic formula. The restriction is therefore not 'per se' a reason to fear incompleteness of the logic.

1. $\vdash \Box p \rightarrow [\top]p$ — arbitrary and specific announcement
2. $\vdash [\top]p \rightarrow (\top \rightarrow p)$ — atomic permanence
3. $\vdash (\top \rightarrow p) \leftrightarrow p$ — propositionally
4. $\vdash \Box p \rightarrow p$ — 1, 2, 3, propositionally
5. $\vdash [\Box p]p \leftrightarrow (\Box p \rightarrow p)$ — atomic permanence
6. $\vdash [\Box p]p$ — 4, 5, propositionally

Finally, we show that a derivation rule for necessitation of $\Box$ is derivable in **APAL**. The proof presents another, very short, example of a derivation. But as the reader might have expected this rule in the proof system, we present the result as a proposition and not as an example. In Proposition 7.4 on page 7 we proved the soundness of this principle.

Proposition 30 Necessitation of arbitrary announcement is derivable in **APAL**.

Proof

1. $\vdash \varphi$ — assumption
2. $\vdash [p]\varphi$ — 1, necessition of announcement; choose $p \notin P_\varphi$
3. $\vdash \Box\varphi$ — 2, deriving arbitrary announcement

$\Box$

4.3 Variants of the rule for deriving arbitrary announcement

We now prove completeness for the logic $APAL$. We do this indirectly, by way of an infinitary variant of the axiomatization **APAL**, that we can show to be complete with respect to the $APAL$ semantics. We apply a technique suggested by Goldblatt [14] using the 'necessity forms' that were introduced in Definition 2 on page 3. Necessity forms are used in the formulation of two variants $R^1(\Box)$ and $R^\omega(\Box)$, now to follow, of the rule $R(\Box)$ ('deriving arbitrary announcement') from system **APAL**.

Definition 31

- From $\varphi \to [\theta][p]\psi$, infer $\varphi \to [\theta]\Box\psi$, where $p \notin (P_\varphi \cup P_\theta \cup P_\psi)$. $R(\Box)$
  (already defined)

- From $\boldsymbol{\varphi}([p]\psi)$, infer $\boldsymbol{\varphi}(\Box\psi)$, where $p \notin P_{\boldsymbol{\varphi}} \cup P_\psi$. $R^1(\Box)$

- From $\boldsymbol{\varphi}([\chi]\psi)$ for all $\chi \in \mathcal{L}_{el}$, infer $\boldsymbol{\varphi}(\Box\psi)$. $R^\omega(\Box)$

Axiomatization $\mathbf{APAL}^\omega$ is the variant of **APAL** with the infinitary rule $R^\omega(\Box)$ instead of $R(\Box)$. Axiomatization $\mathbf{APAL}^1$ is the variant of **APAL** with the different finitary rule $R^1(\Box)$ instead of $R(\Box)$.

Proposition 32 The rules $R^1(\Box)$ and $R^\omega(\Box)$ are sound.

Proof The reader may easily verify that the rule $R^\omega(\Box)$ is sound, as this directly corresponds to the semantics for $\Box$: a formula of the form $\Box\psi$ is valid, if $[\varphi]\psi$ is valid for all epistemic φ. Now replace 'valid' by 'derivable', and observe that the argument can be generalized for other necessity forms than the basic necessity form.

The soundness of rule $R^1(\Box)$ is shown exactly as the soundness of $R(\Box)$: in the soundness proof of $R(\Box)$ it was *only* essential that $\varphi \to [\theta][p]\psi$ and $\varphi \to [\theta]\Box\psi$ were in necessity form. $\Box$

Next, we show in Proposition 34 that every **APAL** theorem is a $\mathbf{APAL}^1$ theorem, and vice versa. That proposition requires a lemma.

Lemma 33 Given a necessity form $\boldsymbol{\varphi}(\sharp)$, there are $\psi, \chi \in \mathcal{L}_{apal}$ such that for all $\theta \in \mathcal{L}_{apal}$:

$$\vdash \boldsymbol{\varphi}(\theta) \text{ iff } \vdash \psi \to [\chi]\theta$$

Proof Let $\boldsymbol{\varphi}(\theta)$ be a theorem. Such an instance of a necessity form $\boldsymbol{\varphi}(\sharp)$ has the following shape: the formula θ is entirely on the right (or, if you wish, entirely on the inside); it is successively bound by, in arbitrary order and arbitrarily often, K_a-operators, announcement operators $[\chi']$, and implicative forms $\chi'' \to \dots$. We can 'rearrange the order of these bindings', so to speak, to get the required form $\psi \to [\chi]\theta$. This, of course, is *still* a necessity form. But a fairly simple one. For these rearrangements it does not matter whether the formula $\boldsymbol{\varphi}(\theta)$ contains *other* logical connectives (or even $\Box$ operators!) that were not used as constructors for the necessity form: these remain bound as they already were. We are only shifting around epistemic operators, announcements, and implications that were used to construct the necessity form and other subformulas remain unchanged.

First we look at all the *public announcement modalities* occuring in $\boldsymbol{\varphi}(\theta)$. Using the reduction axioms for public announcement logic, we can push these modalities inside, past all the other components of the necessity form. To push them past the knowledge operators K_a, we use the reduction axiom 'announcement and knowledge':

$$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$$

To push them past implications, we use the axioms 'announcement and negation' and 'announcement and conjunction'. So now *all* the announcement modalities are 'stacked' on the bottom of the necessity form, right in front of θ. We repeatedly apply the axiom annnouncement composition

$$[\varphi][\psi]\eta \leftrightarrow [\varphi \wedge [\varphi]\psi]\eta$$

so that we can collapse all these announcement modalities into one announcement modality.

We now take care of epistemic modalities. So far, what is left of the necessity form $\boldsymbol{\varphi}(\theta)$ is a sequence of symbols of the forms ($\varphi \rightarrow \ldots$ or $K_a \ldots$, followed by, at the bottom ('at right'), $[\chi]\theta$. We do not yet have the desired form '$\psi \rightarrow [\chi]\theta$' because, for example, the right-hand side of the status quo of our efforts may look like $\ldots K_a[\chi]\theta$. First we get rid of all K_a-modalities in that sort of position: we push all the implication symbols $\rightarrow$ past all the K_a-modalities by using that in the axiomatization **S5** theorems of form $\varphi \rightarrow K_a\psi$ can be transformed into theorems of form $\hat{K}_a\varphi \rightarrow \psi$, and vice versa. From left to right: apply monotonicity of $\hat{K}_a$ to both sides of $\varphi \rightarrow K_a\psi$, getting the theorem $\hat{K}_a\varphi \rightarrow \hat{K}_aK_a\psi$. In $S5$, $\hat{K}_aK_a\psi$ is equivalent to $K_a\psi$, so we get $\hat{K}_a\varphi \rightarrow K_a\psi$. Using veracity for K_a, we get $\hat{K}_a\varphi \rightarrow \psi$. From right to left is similar, except that we now first derive $K_a\hat{K}_a\varphi \rightarrow K_a\psi$ from $\hat{K}_a\varphi \rightarrow \psi$. In this way, we iteratively remove all K_a-modalities in wrong position.

Finally, we take care of implications. We now have a theorem of the form $(\varphi_1 \rightarrow \ldots \rightarrow (\varphi_n \rightarrow [\chi]\theta)\ldots)$. By a number of propositional steps this gives us a theorem of form $\psi \rightarrow [\chi]\theta$, as desired.

Clearly, the argument works both ways, as all axioms applied are equivalences. $\Box$

Proposition 34 ($\mathbf{APAL}^1 = \mathbf{APAL}$) Every $\mathbf{APAL}^1$ theorem is an **APAL** theorem, and vice versa.

Proof Suppose we have a derivation involving an application of $R^1(\Box)$, such that given some $\boldsymbol{\varphi}([p]\psi)$, we infer $\boldsymbol{\varphi}(\Box\psi)$. We can now transform this into a derivation with an application of $R(\Box)$. Apply Lemma 33 to $\boldsymbol{\varphi}([p]\psi)$ for $\theta = [p]\psi$. From the result of form $\varphi \rightarrow [\chi][p]\psi$, we now infer $\varphi \rightarrow [\chi]\Box\psi$ by applying rule $R(\Box)$. Again using Lemma 33, now for $\theta = \Box\psi$, we get a derivation of $\boldsymbol{\varphi}(\Box\psi)$. Repeat this for all applications of $R^1(\Box)$. The resulting derivation does not have a single $R^1(\Box)$ application! The argument works in both directions. $\Box$

Finally, we show that every $\mathbf{APAL}^\omega$ theorem is a $\mathbf{APAL}^1$ theorem.

Proposition 35 ($\mathbf{APAL}^\omega \subseteq \mathbf{APAL}^1$) Every $\mathbf{APAL}^\omega$ theorem is an $\mathbf{APAL}^1$ theorem.

Proof Let us observe that the rule $R^1(\Box)$ is stronger than the rule $R^\omega(\Box)$: if we can prove $\boldsymbol{\varphi}([\theta]\psi)$ for all epistemic formulas θ then we can prove in particular $\boldsymbol{\varphi}([p]\psi)$ for some atom $p \notin P_{\boldsymbol{\varphi}} \cup P_\psi$. As a result, we can derive the conclusion of the infinitary rule using only the finitary rule $R^1(\Box)$, and the axiomatization based on the infinitary rule $R^\omega(\Box)$ defines a set of theorems that is included in or equal to the set of theorems for the axiomatization based on the finitary rule $R^1(\Box)$. $\Box$

4.4 Completeness of the axiomatization $\mathbf{APAL}^\omega$

Let us now demonstrate that the axiomatization based on the infinitary rule $R^\omega(\Box)$ is complete with respect to the semantics. We use Goldblatt's technique applying necessity forms, where the main effect of rule $R^\omega(\Box)$ is that it makes the canonical model (consisting of all maximal consistent sets of formulas closed under the rule) standard for $\Box$.

A set x of formulas is called a *theory* if it satisfies the following conditions:

- x contains the set of all theorems;
- x is closed under the rule of modus ponens and the rule $R^\omega(\Box)$.

Obviously, the least theory is the set of all theorems whereas the greatest theory is the set of all formulas. The latter theory is called the trivial theory. A theory x is said to be consistent if $\bot \notin x$. Let us remark that the only inconsistent theory is the set of all formulas. We shall say that a theory x is maximal if for all formulas φ, $\varphi \in x$ or $\neg\varphi \in x$. Let x be a set of formulas. For all formulas φ, let $x + \varphi = \{\psi \mid \varphi \to \psi \in x\}$. For all agents a, let $K_a x = \{\varphi \mid K_a\varphi \in x\}$. For all formulas φ, let $[\varphi]x = \{\psi \mid [\varphi]\psi \in x\}$.

Lemma 36 Let x be a theory, φ be a formula, and a be an agent. Then $x + \varphi$, $K_a x$ and $[\varphi]x$ are theories. Moreover $x + \varphi$ is consistent iff $\neg\varphi \notin x$.

Proof We will only prove that $K_a x$ is a theory. First, let us prove that $K_a x$ contains the set of all theorems. Let ψ be a theorem. By the necessitation of knowledge, $K_a\psi$ is also a theorem. Since x is a theory, then $K_a\psi \in x$. Therefore, $\psi \in K_a x$. It follows that $K_a x$ contains the set of all theorems. Second, let us prove that $K_a x$ is closed under modus ponens. Let ψ, χ be formulas such that $\psi \in K_a x$ and $\psi \to \chi \in K_a x$. Thus, $K_a\psi \in x$ and $K_a(\psi \to \chi) \in x$. Since $K_a\psi \to (K_a(\psi \to \chi) \to K_a\chi)$ is a theorem and x is a theory, then $K_a\psi \to (K_a(\psi \to \chi) \to K_a\chi) \in x$. Since x is closed under modus ponens, then $K_a\chi \in x$. Hence, $\chi \in K_a x$. It follows that $K_a x$ is closed under modus ponens. Third, let us prove that $K_a x$ is closed under $R^\omega(\Box)$. Let $\boldsymbol{\varphi}$ be a necessity form and ψ be a formula such that $\boldsymbol{\varphi}([\chi]\psi) \in K_a x$ for all $\chi \in \mathcal{L}_{el}$. It follows that $K_a\boldsymbol{\varphi}([\chi]\psi) \in x$ for all $\chi \in \mathcal{L}_{el}$. Since x is a theory, then $K_a\boldsymbol{\varphi}(\Box\psi) \in x$. Consequently, $\boldsymbol{\varphi}(\Box\psi) \in K_a x$. It follows that $K_a x$ is closed under $R^\omega(\Box)$. $\Box$

Lemma 37 (Lindenbaum lemma) Let x be a consistent theory. There exists a maximal consistent theory y such that $x \subseteq y$.

Proof Let $\psi_0, \psi_0, \ldots$ be a list of the set of all formulas. We define a sequence $y_0, y_1, \ldots$ of consistent theories as follows. First, let $y_0 = x$. Second, suppose that, for some $n \geq 0$, y_n is a consistent theory containing x that has been already defined. If $y_n + \psi_n$ is inconsistent and $y_n + \neg\psi_n$ is inconsistent then, by lemma 36, $\neg\psi_n \in y_n$ and $\neg\neg\psi_n \in y_n$. Since $\neg\psi_n \to (\neg\neg\psi_n \to \bot)$ is a theorem, then $\neg\psi_n \to (\neg\neg\psi_n \to \bot) \in y_n$. Since y_n is closed under modus ponens, then $\bot \in y_n$: a contradiction. Hence, either $y_n + \psi_n$ is consistent or $y_n + \neg\psi_n$ is consistent. If $y_n + \psi_n$ is consistent then we define $y_{n+1} = y_n + \psi_n$. Otherwise, $\neg\psi_n \in y_n$ and we consider two cases:

In the first case, we suppose that ψ_n is not a conclusion of $R^\omega(\Box)$. Then, we define $y_{n+1} = y_n$.

In the second case, we suppose that ψ_n is a conclusion of $R^\omega(\Box)$. Let $\boldsymbol{\varphi_1}(\Box\chi_1)$, ..., $\boldsymbol{\varphi_k}(\Box\chi_k)$ be all the representations of ψ_n as a conclusion of $R^\omega(\Box)$. We define the sequence $y_n^0, \ldots, y_n^k$ of consistent theories as follows. First, let $y_n^0 = y_n$. Second, suppose that, for some $i < k$, y_n^i is a consistent theory containing y_n that has been already defined. Then it contains $\neg\boldsymbol{\varphi_i}(\Box\chi_i)$. Since y_n^i is closed under $R^\omega(\Box)$, then there exists a formula $\varphi_i \in \mathcal{L}_{el}$ such that $\boldsymbol{\varphi_i}([\varphi_i]\chi_i)$ is not in y_n^i. Then, we define $y_n^{i+1} = y_n^i + \neg\boldsymbol{\varphi_i}([\varphi_i]\chi_i)$. Now, we put $y_{n+1} = y_n^k$. Finally, we define $y = y_0 \cup y_1 \cup \ldots$. It is straightforward to prove that y is a maximal consistent theory such that $x \subseteq y$. $\Box$

The canonical model of $\mathcal{L}_{apal}$ is the structure $\mathcal{M}_c = (W, \sim, V)$ defined as follows:

- W is the set of all maximal consistent theories;

- For all agents a, $\sim_a$ is the binary relation on W defined by $x \sim_a y$ iff $K_a x = K_a y$;
- For all atoms p, V_p is the subset of W defined by $x \in V_p$ iff $p \in x$.

Note that the relations $\sim_a$ are indeed equivalence relations.

Lemma 38 (Truth lemma) Let φ be a formula in $\mathcal{L}_{apal}$. Then for all maximal consistent theories x and for all finite sequences $\vec{\psi} = \psi_1 \dots \psi_k$ of formulas in $\mathcal{L}_{apal}$ such that $\psi_1 \in x$, $[\psi_1]\psi_2 \in x$, ..., $[\psi_1]\dots[\psi_{k-1}]\psi_k \in x$:

$$\mathcal{M}_c|\vec{\psi}, x \models \varphi \text{ iff } [\psi_1]\dots[\psi_k]\varphi \in x.$$

Proof The proof is by induction on φ. The base case follows from the definition of V. The Boolean cases are trivial. It remains to deal with the modalities.

Case $K_a\varphi$:
If $\mathcal{M}_c|\vec{\psi}, x \not\models K_a\varphi$ then there exists a maximal consistent theory y such that $x \sim_a y$, $\psi_1 \in y$, $[\psi_1]\psi_2 \in y$, ..., $[\psi_1]\dots[\psi_{k-1}]\psi_k \in y$ and $\mathcal{M}_c|\vec{\psi}, y \not\models \varphi$. By induction hypothesis, $[\psi_1]\dots[\psi_k]\varphi \notin y$. Since $x \sim_a y$, then $K_a x = K_a y$. Thus, $K_a[\psi_1]\dots[\psi_k]\varphi \notin x$, and $[\psi_1]\dots[\psi_k]K_a\varphi \notin x$. Reciprocally, if $[\psi_1]\dots[\psi_k]K_a\varphi \notin x$ then $K_a[\psi_1]\dots[\psi_k]\varphi \notin x$. Let $y = K_a x + \neg[\psi_1]\dots[\psi_k]\varphi$. The reader may easily verify that y is a consistent theory. By Lindenbaum lemma, there is a maximal consistent theory z such that $y \subseteq z$. Hence, $K_a x \subseteq z$ and $[\psi_1]\dots[\psi_k]\varphi \notin z$. Consequently, $x \sim_a z$, $\psi_1 \in z$, $[\psi_1]\psi_2 \in z$, ..., $[\psi_1]\dots[\psi_{k-1}]\psi_k \in z$ and, by induction hypothesis, $\mathcal{M}_c|\vec{\psi}, z \not\models \varphi$. Therefore, $\mathcal{M}_c|\vec{\psi}, x \not\models K_a\varphi$.

Case $[\psi]\varphi$:
Let x be a state in the canonical model $\mathcal{M}_c$ and let $\psi_1, \dots, \psi_{k-1}, \psi_k$ be formulas such that $\psi_1 \in x, \dots, [\psi_1]\dots[\psi_{k-1}]\psi_k \in x$. If $\mathcal{M}_c|\vec{\psi}, x \not\models [\psi]\varphi$, then $\mathcal{M}_c|\vec{\psi}, x \models \psi$ and $\mathcal{M}_c|\vec{\psi}|\psi, x \not\models \varphi$. Thus, by induction hypothesis, $[\psi_1]\dots[\psi_{k-1}][\psi_k]\psi \in x$ and $[\psi_1]\dots[\psi_{k-1}][\psi_k][\psi]\varphi \notin x$. Reciprocally, if $[\psi_1]\dots[\psi_{k-1}][\psi_k][\psi]\varphi \notin x$ then $[\psi_1]\dots[\psi_{k-1}][\psi_k]\psi \in x$ and by induction hypothesis, $\mathcal{M}_c|\vec{\psi}|\psi, x \not\models \varphi$. Thus, $\mathcal{M}_c|\vec{\psi}, x \not\models [\psi]\varphi$.

Case $\Box\varphi$:
Let x be a state in the canonical model $\mathcal{M}_c$ and let $\psi_1, \dots, \psi_{k-1}, \psi_k$ be formulas such that $\psi_1 \in x$, ..., $[\psi_1]\dots[\psi_{k-1}]\psi_k \in x$. If $\mathcal{M}_c|\vec{\psi}, x \not\models \Box\varphi$ then there is a $\Box$-free formula ψ such that $\mathcal{M}_c|\vec{\psi}, x \not\models [\psi]\varphi$. Thus, by induction hypothesis, $[\psi_1]\dots[\psi_{k-1}][\psi_k][\psi]\varphi \notin x$. Using the axiom $\Box\varphi \to [\psi]\varphi$ and applying k times the rule of necessitation, this implies that $[\psi_1]\dots[\psi_{k-1}][\psi_k]\Box\varphi \notin x$. Reciprocally, if $[\psi_1]\dots[\psi_{k-1}][\psi_k]\Box\varphi \notin x$ then, using the fact that x is closed with respect to the special inference rule for $\Box$, there is a $\Box$-free formula ψ such that $[\psi_1]\dots[\psi_{k-1}][\psi_k][\psi]\varphi \notin x$. Thus, by induction hypothesis, $\mathcal{M}_c|\vec{\psi}, x \not\models [\psi]\varphi$ and $\mathcal{M}_c|\vec{\psi}, x \not\models \Box\varphi$. □

4.5 Completeness of the axiomatization APAL

As a result we now have completeness for our logic **APAL**.

Proposition 39 (Completeness) Let φ be a formula in $\mathcal{L}_{apal}$. Then φ is a theorem (in **APAL**) if φ is valid.

Proof Let φ be valid. From Lemmas 36, 37, and 38 follows that φ is a theorem in $\mathbf{APAL}^\omega$. From that, Proposition 34, and Proposition 35 follows that φ is a theorem in **APAL**. □

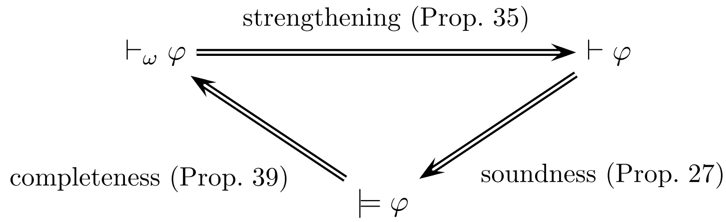


Figure 3: Method to prove soundness and completeness

Theorem 40 (Soundness and completeness) Let φ be a formula in $\mathcal{L}_{apal}$. Then φ is a theorem iff φ is valid.

Proof Soundness was proved in Proposition 27. Completeness was proved in Proposition 39. □

4.6 Further proof theoretical observations

We wish to emphasize the rather peculiar nature of this completeness proof. Given a logic $APAL$, a finitary axiomatization **APAL**, and an infinitary version $\mathbf{APAL}^\omega$ of that axiomatization with the infinitary rule $R^\omega(\Box)$, we have been proceeding as follows. First, we showed that every theorem of **APAL** is a validity of $APAL$ (soundness). Then we showed that every validity of $APAL$ is derivable in $\mathbf{APAL}^\omega$ by a canonical model argument (completeness). Finally we observed that every theorem of $\mathbf{APAL}^\omega$ is also derivable in **APAL**, by two observations. Firstly, an application of $R^\omega(\Box)$ can be adjusted to an application of $R^1(\Box)$: instead of deriving the conclusion with unique occurrence $\Box\psi$ from an infinite set (namely for all epistemic formulas φ) of premises $[\varphi]\psi$, we pick out a premise with a fresh atom among that infinity and derive the conclusion from $[\varphi]\psi$ only. Secondly, the other observation is that we can transform derivations with $R^1(\Box)$ applications into derivations with $R(\Box)$ applications.

The crucial aspect is that the canonical model is for the infinitary version $\mathbf{APAL}^\omega$ of the proof system, and not for the finitary proof system **APAL**. The infinitary version is strongly complete: from Lemma 37 and Lemma 38 follows that every consistent theory is satisfied in a model, one of the formulations of strong completeness. But this does not imply compactness, because the proof system is not finitary.

The finitary proof system **APAL** is only weakly complete: *when proving theorems*, or in other words proofs without premisses, applications of the infinitary rule $R^\omega(\Box)$ can be replaced by applications of the finitary rule $R(\Box)$, and that proof can then be transformed to one using the finitary rule in **APAL**. But in infinitary proofs, starting from infinitely many assumptions, we cannot use this trick without getting rid of our proof assumptions. So strong completeness cannot be shown for the finitary axiomatization **APAL**, and indeed, as we have seen in Proposition 21, $APAL$ is not compact.

5 Arbitrary events

Along a common line in dynamic epistemics, one might consider more general accessibility relations on our structures (as summarily explored in Proposition 20), and one might expand the language with additional modal operators, in particular: with common knowledge, with actions that are not public (such as private announcements), and with assignments (actions that change the truth value of atomic propositions). Let us consider 'arbitrary events' in the sense of arbitrary action models [5].

In public announcement logic all events are public. More complex dynamics is also conceivable, such as private messages, events involving partial observation, etc. Action models formalize such more complex dynamics. These were proposed by Baltag et al. in [5]. We refrain from giving sufficient technical details to understand how these action models work for a reader who has not come across them before, and merely mention that an action model is a structure exactly as a Kripke model except that elements of the domain are called 'events' u instead of 'states' s, and that instead of a valuation V, that for each state determines which facts are true and false, we now have a precondition function pre, that to each event assigns a formula called a *precondition*. This formula is the precondition for the execution of that event. A singleton action model with universal access for all agents corresponds to a public announcement, and the precondition for the event 'public announcement' is the announcement formula.

Let U be a finite action model. Some possible generalizations are as follows ($*$ is arbitrary finite iteration). A sensible restriction in the semantics for arbitrary actions is that all preconditions must be epistemic formulas.

1	$M, s \models \langle U \rangle \varphi$	iff	there is a $u \in U : M, s \models \langle U, u \rangle \varphi$
2	$M, s \models \Diamond \varphi$	iff	$M, s \models \langle U \rangle^* \varphi$ for a given U
3	$M, s \models \Diamond \varphi$	iff	there is a U of a given signature $: M, s \models \langle U \rangle \varphi$
4	$M, s \models \Diamond \varphi$	iff	there is a $U : M, s \models \langle U \rangle \varphi$

In the first two proposals a given action model U is a parameter of the language. The first was investigated by Hoshi in [16, p.8]. The second can be seen as a generalization of iterated relativization which was investigated in [19], and it results in undecidable logics. In the third we allow action models of a given signature, i.e. an action model frame without preconditions for action point execution. The logic $APAL$ comes under this category: it is arbitrary event logic for the signature 'singleton': this sort of action model corresponds to an announcement.

The last proposal seems the endpoint of further generalization. From a multi-agent perspective, where more complex than public events are conceivable, this also seems the most obvious perspective for multi-agent knowability. Note that action model logic (without $\Box$) is again equally expressive as multi-agent logic. All validities in Proposition 7 hold, and we conjecture that CR also holds. Axiom MK does not hold. Even in finite models there are infinite chains of informative actions, because the uncertainty of agents about each other's uncertainty can be arbitrarily complex. An example is, given initial uncertainty of two agents a, b about the value of an atom p, that a privately learns that p, after which b privately learns that a privately learnt that p, after which a privately learns *that*, and so on, thus creating an arbitrarily large finite model satifying $K_a K_b K_a K_b \dots p$ but where b does not know *that*. Proposition 12 stating that the truth of $\Diamond \psi$ can always be simulated by the truth of $\langle p \rangle \psi$ for some fresh atom p, has a natural generalization to replacing formula preconditions in action models by fresh atoms. In the axiomatization **APAL** (Table 2) we have to add the various axioms reducing the postconditions of updates, and we have to replace the axiom 'announcement and knowledge' by its action-model-reducing counterpart

$\Box \varphi \rightarrow [U] \varphi$ where for all events $u \in U : \mathsf{pre}(u) \in \mathcal{L}_{el}$ — arbitrary and specific event

Unfortunately, it is unclear what derivation rule should allow the introduction of a $\Box$-formula;

an inference involving announcement of a fresh atomic variable is certainly not good enough: this atom only 'witnesses' a public announcement, and not other action models.

If factual change is also permitted, one has the peculiar result that $\Diamond\varphi$ is valid for all consistent φ, in other words, all satisfiable formulas are realizable (reachable) in *any information state* (subject to the restriction that the information state is finite). This applies a technical result in van Ditmarsch and Kooi [30]: given two finite information states, there is an event transforming the first into the second. Allowing factual change seems a too drastic departure from the original Fitch question what true formulas are knowable: that seems to suggest that only informative actions are allowed to get to know things, but not factual change.

6 Conclusions and further research

We proposed an extension of public announcement logic with a dynamic modal operator $\Box\varphi$ expressing that φ is true after every announcement ψ. We gave various semantic results, defined fragments of 'knowable' formulas in the Fitch sense that $\models \varphi \rightarrow \Diamond K_a\varphi$, and we showed completeness for a Hilbert-style axiomatization of this logic.

We anticipate a number of further investigations, by us or others. More details on model checking and decidability would be relevant—in particular the somewhat surprising undecidability result. For that, see [11].

Results on model checking and decidability are also relevant for the 'grander scheme' comparing dynamic epistemic logics and temporal epistemic logics, as in recent work by van Benthem et al. [28], and in work in progress by Hoshi [17]. In the comparison between temporal with dynamic epistemics, if we let an announcement correspond to a tick of the clock, a dynamic announcement operator $[\varphi]$ therefore corresponds to a temporal 'next' operator, and our arbitrary announcement operator $\Diamond$ then corresponds to the temporal future operator F, for 'some time in the future'.

Given the proved validities for $\Box$, a relevant question seems where in the S4 scheme of logics the logic $APAL$ resides. It is not S5, but at least (given CR and MK) S4.1 and S4.2. Unfortunately it is unclear whether $\Box$ is a a normal modal operator, more concretely: whether the schematic validities (i.e., those employing formula variables, such as $\Box\varphi \rightarrow \Box\Box\varphi$, not those employing propositional variables, such as $\Box p \leftrightarrow p$) in $\mathcal{L}_\Box \cap APAL$ satisfy uniform substitution. Tentative evidence against it, is that public announcement logic is not normal, e.g. $[p]p$ is valid but $[p \wedge \neg Kp](p \wedge \neg Kp)$ is invalid. Further tentative evidence against normality is that an interpretation of $\Box$ in terms of neighbourhood semantics is conceivable [2]; which points to non-normality. On the other hand, arbitrary announcement logic *with* $\Box$ but *without* announcements might just as well be equally expressive as $APAL$. This is the logic with language $\varphi ::= p|\neg\varphi|\varphi \wedge \varphi|K_a\varphi|\Box\varphi$ and with $\Box$-semantics: $M, s \models \Box\varphi$ iff for all $\psi \in \mathcal{L}_{el} : M, s \models \psi$ implies $M|\psi, s \models \varphi$. If so, that would be suggestive evidence for normality of $\Box$.

Because of these uncertainties about the character of $\Box$ it is sometimes difficult to interpret our results. For example, the principle MK ($\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$) in conjunction with 4 ($\Box\varphi \rightarrow \Box\Box\varphi$) correspond to the frame property of *atomicity*, defined as $\forall x \exists y(Rxy \wedge \forall z(Ryz \rightarrow z = y))$ (see [6, p.167, Ex.3.57]). In our terms, atomicity seems to describe that one can always make a *most* informative announcement. But this is false! Consider the model consisting of $2^{|P|}$ states namely one for each valuation of atomic propositions; and with universal access on the domain for all agents. Every given epistemic formula contains only a finite number of atoms, so after its announcement a further informative announcement remains possible. So a most informative announcement can not always be made. This puzzles us.

Our $\Box$-operator is an implicit quantification over announcements. Of course, one can also make the quantification explicit. In other words, instead of $\Diamond\varphi$ we may as well write $\exists\psi\langle\psi\rangle\varphi$. This approach is currently investigated by Baltag.

Unlike public announcement logic, arbitrary announcement logic can also be used to specify *planning problems*, as in AI: we can express some initial knowledge conditions, and a final desideratum in terms of knowledge, and a diamond $\Diamond$ of unknown instantiation representing a sequence of announcements supposedly realizing it. In other words, something of the form $\mathsf{init} \rightarrow \Diamond K\mathsf{final}$. Different variants of this theme are conceivable. If our logic 'works', we can reduce and manipulate such an expression so that it should ultimately deliver the concrete announcements needed to realize the final knowledge conditions: a plan. We did not pursue this matter although a tableaux calculus for $APAL$ may be relevant to mention here [4].

Acknowledgements

This research started late August 2005 when Andreas, Tiago, and Hans were brainstorming in Andreas' office on quantifying over announcements in order to devise a logic of planning. Thus the $\Box$ came up during the discussion as a way to express what is true after arbitrary announcements. Hans thinks Tiago came up with the idea, whereas Andreas thinks that Hans came up with the idea, and so on. As so often, it seems to have come up in a true spirit of mutual and free exchange of ideas and collaboration. Over time, others joined the collaboration. Six authors, six nationalities, only one of those (Philippe) working in his country of origin. Incidentally, the intended 'logic of planning' never materialized. We gratefully acknowledge various input from many people. We put them in alphabetical order. Even so, Johan van Benthem deserves to be thanked separately: sometime in 2006 he pointed us towards his work on knowability and also gave detailed comments on the status quo of our research. For their input, we thank: Nick Asher, Johan van Benthem, Balder ten Cate, Jan van Eijck, Tim French, Amelie Gheerbrand, Barteld Kooi, Ron van der Meijden, Larry Moss, Dung Nguyen, Eric Pacuit, Greg Restall, Joe Salerno, Hartley Slater, Yde Venema, Rineke Verbrugge, and Albert Visser. Finally, we wish to thank the RSL reviewer for the praise heaped on the paper.

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