A Bochev–Dohrmann–Gunzburger stabilization method for the primitive equations of the ocean

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A R T I C L E   I N F O

Article history:
Received 5 September 2012
Received in revised form 26 October 2012
Accepted 26 October 2012

Keywords:
Primitive equations
Finite elements
Stabilized methods
Inf–sup condition

A B S T R A C T

We introduce a low-order stabilized discretization of the primitive equations of the ocean with highly reduced computational complexity. We prove stability through a specific inf–sup condition, and weak convergence to a weak solution. We also perform some numerical tests for relevant flows.

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doi:10.1016/j.aml.2012.10.015

1. Introduction

The primitive equations (PEs) of the ocean are a mathematical model for large space and time scales of oceanic flow and are extensively used for climatic, weather and ecological studies [1–3]. The existence of weak solutions \((u, p)\) (with \(H^1 \times L^2\) regularity) has been proved [4,5], as well as the existence and uniqueness of strong solutions (with \(H^2 \times H^1\) regularity) [6,7]. Finite element discretizations are well suited to irregular oceanic bottoms. In this paper we introduce a stabilized discretization of the PEs for first-order finite elements. We adapt the Bochev–Dohrmann–Gunzburger stabilization technique [8] to a reduced model of PEs that retains only the (3D) horizontal velocity and the (2D) surface pressure as unknowns. This yields a solver with highly reduced computational complexity. We introduce the reduced PE model in Section 2 and the numerical discretization in Section 3. We prove the stability and convergence of the discretization based on a specific inf–sup condition in Section 4. In Section 5 we describe some numerical tests for relevant flows.

2. Primitive equations of the ocean

Let \(\omega\) be a bounded domain in \(\mathbb{R}^{d-1}\) \((d = 2 \text{ or } d = 3)\) that represents a piece of the ocean surface, and let \(D: \omega \to \mathbb{R}\) be a depth function. We consider the ocean domain \(\Omega = \{(x, z) \in \mathbb{R}^d \mid x \in \omega, -D(x) \leq z \leq 0\}\). For simplicity we assume that \(\omega\) is polygonal and \(D\) is piecewise affine on some triangulation of \(\omega\), so that \(\Omega\) is a polyhedron with a flat surface. We suppose that the boundary of \(\Omega\) is split as \(\partial \Omega = \Gamma_s \cup \Gamma_b\), where \(\Gamma_s = \{(x, 0) \in \mathbb{R}^d; x \in \omega\}\) represents the
ocean surface and \( \Gamma_b = \partial \Omega - \Gamma_s \), the ocean bottom and, eventually, sidewalls. We consider the following steady reduced PE model.

Find a horizontal velocity field \( \mathbf{u} : \overline{\Omega} \mapsto \mathbb{R}^{d-1} \) and a surface pressure \( p : \overline{\Omega} \mapsto \mathbb{R} \) such that

\[
\begin{aligned}
\begin{cases}
(u, u_z) \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla_H p + \mathbf{u}^\perp = \mathbf{f}, \\
\nabla_H \cdot (\int_{\Gamma_s} \mathbf{u} \cdot \mathbf{s} \, d\Gamma) = 0 \\in \omega,
\end{cases}
\end{aligned}
\]

\[ \mathbf{u}|_{\Gamma_b} = 0, \quad \mu \partial_z \mathbf{u}|_{\Gamma_b} = \tau, \]

where \( \mu \) is the viscosity coefficient and \( \nabla_H = (\partial_x, -\partial_y) \) denotes the horizontal gradient. The term \( \mathbf{u}^\perp \) represents the Coriolis acceleration, which only appears when \( d = 3 \). In this case, if \( \mathbf{u} = (u_1, u_2, 0) \), then \( \mathbf{u}^\perp = (-u_2, u_1) \). Thus, we define \( \varphi = 0 \) for \( d = 2 \) and \( \varphi = 2 \theta \sin \phi \), where \( \theta \) is the angular rotation rate of the earth and \( \phi \) is latitude, for \( d = 3 \). The source term \( \mathbf{f} \) takes into account variable density effects due to variations in temperature and salinity and \( \tau \) is the wind tension at the surface.

This model is an approximation of the Navier–Stokes equations for thin domains [9]. In particular, the pressure is assumed to be hydrostatic. The surface pressure \( p \) may be interpreted as the pressure that must be exerted at the flow surface to keep it flat. It is the Lagrange multiplier associated with the second equation in (1), which represents mass conservation. Observe that the 3D velocity field \((\mathbf{u}, u_z)\) is incompressible and that \( u_z = 0 \) on \( \Gamma_b \). This is the rigid lid assumption.

Consider the following spaces for the velocities and pressures:

\[
\begin{aligned}
\mathbf{W}_b^{1,k}(\Omega) &= \{ \mathbf{v} \in \mathbf{W}^{1,k}(\Omega)^{d-1} : \mathbf{v}|_{\Gamma_b} = 0 \}, \\
\mathbf{H}^1_b(\Omega) &= \mathbf{W}^{1,2}_b(\Omega), \\
\mathbf{L}^0_b(\omega) &= \left\{ q : \omega \mapsto \mathbb{R} \text{ measurable such that } \int_{\omega} D(\mathbf{x})|q(\mathbf{x})| \, dx < \infty \right\}, \\
\mathbf{L}^r_{D,0}(\omega) &= \mathbf{L}^r_b(\omega)/\mathbb{R}.
\end{aligned}
\]

We define the weak solutions of (1) as the solutions of the following variational formulation.

Given \( \mathbf{f} \in \mathbf{H}^1_b(\Omega)' \) and \( \tau \in \mathbf{H}^{-1/2}(\Gamma_b) \), find \((\mathbf{u}, p) \in \mathbf{H}^1_b(\Omega) \times \mathbf{L}^{3/2}_{D,0}(\omega) \) such that

\[
\begin{aligned}
B(\mathbf{u}; (\mathbf{u}, p), (\mathbf{v}, q)) &= L(\mathbf{v}), \
\forall (\mathbf{v}, q) \in \mathbf{W}^{1,3}_b(\Omega) \times \mathbf{L}^2_{D,0}(\omega), \quad \text{where} \\
B(\mathbf{a}; (\mathbf{u}, p), (\mathbf{v}, q)) &= (\mathbf{a} \cdot \nabla \mathbf{u}, \mathbf{v}) + \mu (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla_H \cdot \mathbf{v}) + (\nabla_H \cdot \mathbf{u}, q) + (\mathbf{u}^\perp, \mathbf{v}), \\
L(\mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega} + \langle \tau, \mathbf{v} \rangle_{\Gamma_b},
\end{aligned}
\]

where \( \mathbf{a} = (a_1, a_2) \) for some \( \mathbf{a} \in \mathbf{H}^1_b(\Omega) \), with \( a_2 \) defined from \( \mathbf{a} \) as in (1). The convection term is defined by duality as \( (\mathbf{a} \cdot \nabla \mathbf{u}, \mathbf{v}) = -\int_{\Omega} (\mathbf{a} \cdot \nabla \mathbf{v}) \mathbf{u} \). This Petrov–Galerkin formulation is needed when \( d = 3 \) (not when \( d = 2 \) because the vertical velocity \( a_1 \) has only \( L^2 \) regularity, and then the convection operator does not have \( H^{-1} \) regularity. Problem (2) was investigated in a previous study [5].

3. Numerical scheme

Consider a family of triangulations \( \{ \mathcal{T}_h \}_{h>0} \) of \( \Omega \). For each \( T \in \mathcal{T}_h \) we define the prism \( P_T = \{(x, z) \in \mathbb{R}^d : (x, z) \in T \} \), such that \( x \in T, -D(x) \leq z \leq 0 \). Consider a triangulation \( \mathcal{T}_h \) of \( \Omega \) associated with \( \mathcal{C}_h \) by subdividing each prism \( P_T \) into triangles (when \( d = 2 \)) or tetrahedra (when \( d = 3 \)) in such a way that the projection of any \( K \in \mathcal{T}_h \) on \( \Gamma_b \) (that we identify with \( \omega \)) is an element of \( \mathcal{C}_h \). Consider the finite-element spaces \( \mathbf{V}_h = \{ \mathbf{v}_h \in C^0(\overline{\Omega})^{d-1} : \mathbf{v}_h|_{K} \in P_1(K)^{d-1}, \forall K \in \mathcal{T}_h, \mathbf{v}_h|_{\Gamma_b} = 0 \}, \mathbf{Q}_h = \{ q_h \in C^0(\overline{\Omega}) : q_h|_{T} \in P_0(T), \forall T \in \mathcal{C}_h \}, \mathbf{R}_h = \{ \phi \in L^2(\omega) : \phi|_{T} \in P_0(T), \forall T \in \mathcal{C}_h \}, \mathbf{P}_m(K) \) the space of polynomials on \( K \) of degree less than or equal to \( m \) and similarly \( P_m(T) \). For all \( T \in \mathcal{C}_h \), we denote by \( b_T \) the barycenter of \( T \) and we define the interpolation operator \( I^\mathcal{C}_h : C^0(\overline{\Omega}) \mapsto \mathbf{V}_h \) such that \( I^\mathcal{C}_h \phi|_T = \phi(b_T), \forall T \in \mathcal{C}_h \). We denote \( I^\mathcal{C}_h \) as \( Id - I^\mathcal{C}_h \).

We discretize (2) as follows. Find \((\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathbf{P}_h \) such that

\[
\begin{aligned}
B_h(\mathbf{u}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) &= L(\mathbf{v}_h), \
\forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times \mathbf{P}_h,
\end{aligned}
\]

where \( B_h(\mathbf{a}; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = B(\mathbf{a}; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) + s_h(p_h, q_h) \). Here, the stabilization term \( s_h \) is defined as

\[
s_h(p_h, q_h) = h^2(D \Pi_{\sigma} p_h, \Pi_{\sigma} q_h)_\omega \quad \text{with } \sigma = 0 \text{ if } d = 2 \text{ and } \sigma = 1 \text{ if } d = 3.
\]

The analysis that follows shows that the term \( s_h \) yields the stability of the pressure discretization in the natural norms associated with (2).

4. Stability and convergence analysis

The stability of discretization (3) follows from the following discrete inf–sup condition.
Lemma 1. Assume that the family of triangulations \( \{ T_h \}_{h > 0} \) is uniformly regular. Then for any \( r \in (1, +\infty) \) there exists a constant \( \gamma_r > 0 \) independent of \( h \) such that \( \forall q_h \in P_h \),

\[
\gamma_r \| q_h \|_{r,0,0} \leq \sup_{v_h \in \mathcal{U}_h - \{ 0 \}} \frac{(\nabla H \cdot v_h, q_h)}{\| v_h \|_{1,1,\Omega}} + h^{\frac{d}{2} - \frac{d}{r}} \| \Pi_h q_h \|_{r,0,0}, \quad \text{where } \frac{1}{s} + \frac{1}{r} = 1.
\]  

(5)

Proof. Given \( q_h \in P_h \), we consider its extension \( \tilde{q}_h \) to \( \overline{\Omega} \) defined by \( \tilde{q}_h(x, z) = q_h(x) \), \( \forall x \in \omega, -D < z \leq 0 \). By Amrouche and Girault [10], there exists a constant \( \gamma_0 > 0 \) such that \( \gamma_0 \| q_h \|_{0,0,\omega} \leq \sup_{v \in [W_0^{1,2}(\omega)]^d - \{ 0 \}} (\nabla \cdot v, \tilde{q}_h) / \| v \|_{1,1,\omega} \). If we denote \( \tilde{v} = (v, v_z) \), observe that \( (\partial_z v_z, \tilde{q}_h) = 0 \), because \( \partial_z \tilde{q}_h = 0 \) and \( v_z = 0 \) in \( \partial \omega \). As \( \| q_h \|_{0,0,\omega} = \| q_h \|_{0,0,\omega} \), it follows that \( \gamma_0 \| q_h \|_{0,0,\omega} \leq \sup_{v \in [W_0^{1,2}(\omega)]^d - \{ 0 \}} (\nabla \cdot v, \tilde{q}_h) / \| v \|_{1,1,\omega} \). Therefore, there exists \( v \in [W_0^{1,2}(\omega)]^d - \{ 0 \} \) such that

\[
\gamma_0 \| q_h \|_{0,0,\omega} \leq \frac{(\nabla H \cdot v, \tilde{q}_h)}{\| v \|_{1,1,\omega}}.
\]  

(6)

We use an adaptation of Verfürth's trick [11]. There exists \( v, v_h \in \mathcal{U}_h \cap [H_0^1(\Omega)]^d - \{ 0 \} \) such that

\[
\| v \|_{1,1,\omega} \leq c \| v \|_{1,1,\omega}, \quad \| v - v_h \|_{0,0,K} \leq c h_K \| v \|_{1,1,K}
\]  

(7)

for some constant \( c \) independent of \( h \). Using the first inequality in (7), we obtain

\[
\frac{(\nabla H \cdot v, \tilde{q}_h)}{\| v \|_{1,1,\omega}} \leq c \frac{(\nabla H \cdot v_h, \tilde{q}_h)}{\| v_h \|_{1,1,\omega}} + c \frac{(v - v_h, \tilde{q}_h)}{\| v \|_{1,1,\omega}}.
\]  

(8)

Then

\[
(\nabla H \cdot v, \tilde{q}_h) = -(v - v_h, \nabla H \tilde{q}_h) \leq \left( \sum_{k \in \mathcal{T}_h, 0} \| v - v_h \|_{0,0,K} h_K^{\frac{1}{r}} \right)^{1/r} \left( \sum_{k \in \mathcal{T}_h, 0} h_K^r \| \nabla H \tilde{q}_h \|_{0,0} \right)^{1/r} \leq c \| v \|_{1,1,\omega} h \| \nabla H \tilde{q}_h \|_{0,0,\omega}.
\]  

(9)

Consider the finite-element space \( \tilde{\mathcal{T}}_h = \{ \phi \in L^2(\omega) : \phi |_K \in P_0(K), \forall K \in \mathcal{T}_h \} \). We define the interpolation operator \( \Pi_h : C_0^\infty(\omega) \rightarrow \tilde{\mathcal{T}}_h \) by \( \Pi_h \phi |_K = \phi |_K \in P_0(K) \), \( \forall K \in \mathcal{T}_h \), where \( b_{r,K} \) is some node located in \( K \) whose projection on \( \ell_r \) is \( b_r \). Then

\[
\| (ld - \Pi_h) q_h \|_{1,1,\omega}^2 = \sum_{r \in \mathcal{E}_h} \sum_{K \in \mathcal{T}_h} \int_{\ell_r} |q_h(x, z) - q_h(b_r,K)|^2 \, dx \, dz = \sum_{r \in \mathcal{E}_h} \int_{\ell_r} (D(x))q_h(x) - q_h(b_r,K))^2 \, dx = \|(ld - \Pi_h) q_h \|_{r,0,\omega}^2.
\]  

Using an inverse inequality between polynomial spaces [12] and the regularity of the grids, we have

\[
\| \nabla H \tilde{q}_h \|_{r,0,\omega} = \sum_{k \in \mathcal{T}_h} \| \nabla H \tilde{q}_h - \Pi_h \tilde{q}_h \|_{0,0,K} \leq c_i \sum_{k \in \mathcal{T}_h} h_K^{r(1 - \frac{d}{2} - \frac{d}{r})} \| \tilde{q}_h - \Pi_h \tilde{q}_h \|_{0,0,K} \leq c_i h^{r(1 - \frac{d}{2} - \frac{d}{r})} \| (ld - \Pi_h) q_h \|_{r,0,\omega}^2 = c_i h^{r(1 - \frac{d}{2} - \frac{d}{r})} \| (ld - \Pi_h) q_h \|_{r,0,\omega}^2.
\]  

(10)

Then (5) follows from (9). \( \square \)

We next prove the stability of the discretization (3).

Theorem 1. Assume that the family of grids \( \{ T_h \}_{h > 0} \) is uniformly regular. Then the discrete problem (3) admits a solution \( (u_h, p_h) \in \mathcal{U}_h \times P_h \) that is bounded in \( H^{1}_0(\Omega) \times L^{r,0,\omega}_r \) satisfying

\[
\| u_h \|_{1,1,\omega} \leq \frac{1}{\mu l}, \quad h^{\frac{d}{2}} \| \Pi_h p_h \|_{L^{r,0,\omega}} \leq \frac{1}{\mu l}; \quad \| p_h \|_{r,0,0} \leq \mu \left( \frac{1}{\mu^2} + 1 + \frac{1}{\mu^2} \right),
\]  

(10)

where \( C \) is a constant independent of \( h, l = \| L \|_{-1,1,\omega} \), \( \sigma \) is defined in (4) and \( r = 2 \) when \( d = 2 \) or \( r = \frac{3}{2} \) when \( d = 3 \).

Proof. The existence of solutions of (3) follows from a standard compactness argument in finite dimension lying on the linearization of the convection term. The basis for this proof is estimate (10), whose deduction we describe next. Assume that \( (u_h, p_h) \) is a solution of this problem. Set \( v_h = u_h, q_h = p_h \) in (3) and denote \( \bar{u}_h = (u_h, u_{h2}) \). Then, since \( \nabla \cdot \bar{u}_h = 0 \) and \( u_{h2} \parallel \gamma = 0 \), we have \( \bar{B}_h(\bar{u}_h; (u_h, p_h); (u_h, p_h)) = \mu \| \nabla \bar{u}_h \|_{0,0,\omega} + s_h(p_h, p_h) \). Thus, \( \frac{1}{2} \| u_h \|_{1,1,\omega}^2 + h^{\frac{d}{2}} \| \Pi_h p_h \|_{L^{r,0,\omega}}^2 \leq \frac{1}{2} \| L \|^2 \). This yields the two first estimates in (10). To estimate the pressure we use inf–sup condition (5). Taking \( q_h = 0 \) in (3) and using Sobolev injections and the two first estimates in (10), we have

\[
(\nabla H \cdot v_h, p_h) = (\bar{u}_h \cdot \nabla \bar{u}_h, v_h) + (\mu \nabla u_h, \nabla v) + (\psi \nabla u_h, -L(v_h)) \leq C \left( \| u_h \|_{1,1,\omega}^2 + \mu \| u_h \|_{1,1,\omega} + \| \psi \|_{0,0,\omega} \| u_h \|_{1,1,\omega} + \| L \|_{-1,\omega} \right) \| v_h \|_{1,1,\omega}
\]

\[
\leq C \left( \| u_h \|_{1,1,\omega}^2 + \mu \| u_h \|_{1,1,\omega} + \| \psi \|_{0,0,\omega} \| u_h \|_{1,1,\omega} + \| L \|_{-1,\omega} \right) \| v_h \|_{1,1,\omega}
\]

As the second summand in (5) is estimated in (10), we obtain the pressure estimate in (10). \( \square \)
We finally prove the convergence of discretization (3).

**Theorem 2.** Assume that the family of grids \( \{ T_h \}_{h>0} \) is uniformly regular. Then the sequence \( \{(u_h, p_h)\}_{h>0} \) of solutions of discrete problem (3) contains a subsequence that is weakly convergent in \( H^1_s(\Omega) \times L^2_s(\omega) \) (with \( r \) as in Theorem 1) to a solution of the continuous problem (2).

**Proof.** By Theorem 1, the sequence \( \{(u_h, p_h)\}_{h>0} \) is bounded in \( H^1_s(\Omega) \times L^2_s(\omega) \), which is a reflexive space. Then it contains a subsequence, that we still denote in the same way, that is weakly convergent in that space to a pair \((u, p)\). Consider a pair of test functions \((v, q)\) bounded in \( W^{1,3}_h(\Omega) \times L^2_s(\omega) \). By the interpolation theory for finite elements [12] there exists a sequence \( \{(u_h, q_h)\}_{h>0} \) in \( U_h \times P_h \) that is strongly convergent to \((v, q)\) in \( W^{1,3}_h(\Omega) \times L^2(\omega) \) and in \( W^{1,3}_h(\Omega) \times L^2_s(\omega) \), as \( \| q_h - q \|_{L^2_s(\omega)} \leq \| q_h - q \|_{L^2_s(\omega)} \leq \| q_h - q \|_{L^2_s(\omega)} \leq \| D \|^{1/2}_{0,\infty,w} \| q_h - q \|_{0,\infty,w} \). Moreover, \( \int_{-D(x)}^0 v_h(x, s) ds \) strongly converges to \( \int_{-D(x)}^0 v(x, s) ds \) in \( W^{1,3}_h(\Omega) \). All these convergence results allow us to pass to the limit in all terms of (3) [13]. This proves that \( \lim_{h \to 0} B(u; (u_h, p_h), (v_h, q_h)) = B(u; (u, p), (v, q)) \) to analyze the convergence of the stabilization term, let \( q \in D(\omega) \). We may suppose that \( q_h \) strongly converges to \( q \) in \( L^\infty(\omega) \). Then \( \| \Pi_h q \|_{L^2(\omega)} \leq C h^r/2 \).

Thus, \( \lim_{h \to 0} s_h(p_h, q_h) = 0 \). We deduce that the limit \((u, p)\) is a solution of the continuous problem (2) with test functions \((v, q)\) in \( W^{1,3}_h(\Omega) \times D(\omega) \). As \( D(\omega) \) is dense in \( L^2_s(\omega) \), this holds for all \( q \in L^2_s(\omega) \). If the solution is strong, then it is unique [6] and the whole sequence converges to it by a standard compactness argument. Furthermore, in this case \((u, p) \in H^1(\Omega)^{n-1} \times H^1(\omega)\), and then \((u, p)\) may be taken as test function in problem (2). Then \( \lim_{h \to 0} \| \nabla u_h \|_{2,\infty,\Omega} = \| \nabla u \|_{2,\infty,\Omega} \) and the convergence is strong. A standard argument using the inf–sup condition also proves that the pressures \( p_h \) strongly converge to \( p \) in \( L^2(\omega) \).

We assumed in our analysis that the grids are uniformly regular for brevity. This is not an essential hypothesis and it may be dropped if the discrete inf–sup condition (5) is changed to a more general condition for standard regular grids. This issue is addressed in a forthcoming paper. \( \square \)

### 5. Numerical tests

We solved the 3D steady PEs (1) as the steady state of the evolution equations using a semi-implicit Euler method. Let \( u_0^h = 0 \). For \( n \geq 0 \), given \( u^n_h \in U_h \), find \( u^{n+1}_h, p^{n+1}_h \in U_h \times P_h \) such that,

\[
\frac{1}{\Delta t}(u^{n+1}_h, v_h) + B_h(u^{n+1}_h, p^{n+1}_h, (u^n_h, q_h)) = L(v_h) + \frac{1}{\Delta t}(u^n_h, v_h).
\]

This problem was solved using the application FreeFem++ [14].

**Test 1: Convergence rate.** We set \( \Omega = (0, 1)^3 \), \( \mu = 0.5 \) and the source terms \( f \) and \( \tau \) such that \( p = \exp(x+y) = 2.95 \), \( u = ((2x(z-1) + z^2)^2)(x-1)y(y-1) \), \( (2x(z-1) + z^2)(x-1)^2(y-1)^2(y-1) \). Table 1 shows the estimated convergence orders for the horizontal velocity (in \( H^1(\Omega) \) norm) and surface pressure (in \( L^2(\omega) \) norm) using unstructured regular grids. We recover first order for pressure and somewhat higher orders for velocity that decrease as \( h \) tends to zero.

**Test 2: Upwelling flow.** In this case we consider a swimming-pool domain \( \omega \times (-D(x), 0) \) shown in Fig. 1, where \( \omega = (0, 10000) \times (0, 5000) \) and \( D(x) = \begin{cases} 50 & \text{if } 0 \leq x \leq 4000 \\ 0.05x - 150 & \text{if } 4000 \leq x \leq 5000 \\ 100 & \text{if } 5000 \leq x \leq 10000 \end{cases} \).

We set \( \mu = 10^{-2}, \mu_s = 10^{-7} \text{ m/s}, f = 0 \) and \( \tau = \alpha \psi |\psi|, \) with \( \alpha = 9.27 \cdot 10^{-7} \) and \( \psi = (7.5, 0) \text{ m/s}, \) and \( \varphi = 2 \theta \sin 45^\circ \). In our results the velocity at the surface points \( \pi/4 \) to the right of the wind, according to the Eckman theory (Fig. 2, left). In addition, the pressure increases in the direction of the wind and to its right due to the Coriolis force (Fig. 2, right). Fig. 3 shows a span-wise recirculation induced by the wind and the upwelling and downwellings in a cross-wind plane induced by the interaction between wind and Coriolis forces. All these effects agree with the physics of the flow and with previous numerical results [15].

\[ \text{Table 1} \]

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<td>0.00051</td>
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</table>
Fig. 1. Domain and grid for Test 2.

Fig. 2. Surface horizontal velocity and pressure.

Fig. 3. Flow velocity on the planes $y = 2500$ (left) and $x = 6000$ (right).

Acknowledgments

This research was supported in part by the Spanish Government and FEDER grant MTM2009-07719.

References


