A high order term-by-term stabilization solver for incompressible flow problems

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In this paper, we introduce a low-cost, high-order stabilized method for the numerical solution of incompressible flow problems. This is a particular type of projection-stabilized method where each targeted operator, such as the pressure gradient or the convection, is stabilized by least-squares terms added to the Galerkin formulation. The main methodological originality is that we replace the projection-stabilized structure by an interpolation-stabilized structure, with reduced computational cost for some choices of the interpolation operator. This stabilization has one level, in the sense that it is defined on a single mesh.

We prove the stability of our formulation by means of a specific inf–sup condition, which is the main technical innovation of our paper. We perform a convergence and error estimates analysis, proving the optimal order of accuracy of our method. We include some numerical tests that confirm our theoretical expectations.

Keywords: Finite Elements, Stabilized Methods, High Order Approximation.

1. Introduction

This paper deals with the numerical solution of incompressible flows by low-cost, accurate solvers. This is a challenging issue, as the numerical solution of incompressible flows faces severe stability restrictions. These restrictions essentially concern discretization of the pressure and discretization of dominating convection terms. Eventually, some other dominant terms may be present in the flow equations (e.g., Coriolis forces in geophysical flow equations), which also need a specific treatment in order to be stably discretized.

It is well established that these stability restrictions are treated by limiting a convenient range of high-frequency components of the terms to be stabilized. This is achieved either by enriching the velocity discretization space (Mixed methods), or by adding specific terms to the standard Galerkin discretizations (Stabilized methods). This second procedure may be interpreted as an augmented Galerkin method constructed with an enriched velocity space, via bubble finite element functions (cf. Chacón Rebollo,
However, mixed methods introduce stabilizing degrees of freedom that do not yield accuracy, and thus are more costly than stabilized methods. For this reason, we focus in this paper on stabilized methods.

There exist two classes of stabilized methods: residual-based methods and penalty methods. The first ones yield high-order discretizations and are strongly consistent, but are more expensive than the second ones as they include more terms in the discrete problem. Therefore, in this paper, we follow the second strategy.

Stabilized penalty methods were introduced by Brezzi & Pitkäranta (1984). In their work, to stabilize the pressure discretization, these authors penalize the Galerkin formulation by a pressure laplacian. This method was extended to the term-by-term stabilized method introduced by Chacón Rebollo (1998). In this work, a least-squares penalty term is used to stabilize each actual operator term that needs stabilization. Since these methods are pure penalty methods, their accuracy is limited to first order in the $H^1$ norm for the velocity and in the $L^2$ norm for the pressure, for both Oseen and Navier–Stokes equations.

The projection–stabilization technique designs a high-order penalty stabilized method. This strategy was introduced by Blasco & Codina (2000). In this method, the Galerkin formulation is enriched with a projection penalty term that acts on the pressure gradient: Only high frequencies of the pressure gradient that are not representable in the velocity space are stabilized. This allows one to obtain optimal orders of accuracy. The drawback of this method is that the corresponding projection operator is the $L^2$ projection whose computation is somewhat costly. The local projection–stabilization methods overcome this difficulty by using element-wise projections satisfying suitable orthogonality properties, instead of a global projection operator. Without being exhaustive, among the many references on different versions of local projection–stabilization, let us quote Braack & Burman (2000), Ganesan et al. (2008), Braack et al. (2007), Matthies et al. (2007), Roos et al. (2008) and Knobloch (2010). In particular, in Knobloch (2010) a modification of the local-projection stabilization is introduced for the convection–diffusion equations. In this method, the local projection operator acts on a space of polynomials of one degree smaller than the unknown.

A special class of projection–stabilization methods are the edge-stabilized, also called interior penalty methods (cf. Burman et al., 2006). In these methods, stabilization is achieved by introducing inter-element jumps of the terms to be stabilized. It is proved that it is equivalent to a projection–stabilized method, where the $L^2$ projection operator is replaced by the Oswald (cf. 1993) quasi-interpolant operator on the discrete velocity space. This method has been applied to the solution of Navier–Stokes equations for low- and high-Reynolds numbers, with optimal levels of numerical diffusion (cf. Burman, 2007; Burman & Fernández, 2007).

An alternative to the projection–stabilization method is the subgrid viscosity method introduced by Guermond (1999) and Guermond et al. (2006). This method adds a subgrid viscosity penalty term to the Galerkin finite element discretization that stabilizes the gradient (or the convection) of a range of high frequencies of the unknown and produces a convergence method even for pure transport equations. It differs from standard projection–stabilization methods because it acts directly on the function itself instead of acting on the operator.

In the present paper, we introduce a term-by-term interpolation-stabilized method, with increased accuracy with respect to the penalty term-by-term stabilized method. To our knowledge, it differs from the schemes described in the above references because it uses continuous buffer functions, it does not need enriched finite element spaces, it does not need a projection, and it does not need different meshes. The stabilization procedure is similar to the one introduced in Knobloch (2010), but here the quasi-interpolant acts on the pressure as well as on the convection. The main innovation is that we use a quasi-local approximation operator (or quasi-interpolant) instead of a projection operator. This operator takes
its values in a buffer space, different from the discrete velocity space, but defined on the same mesh, as in Knobloch (2010). This buffer space is a finite element subspace of functions of $H^1(\Omega)^d$ that in each element are standard polynomials with one degree less than the finite element space for velocities. This gives rise to a method with a reduced computational cost for some quasi-interpolant operators, and the same high-order accuracy with respect to projection-stabilized methods. More precisely, the approximation operator is any stable quasi-interpolation operator acting on the test space, and not just an $L^2$ projection operator. This operator could be the Girault & Lions (2001), Bernardi et al. (2004), Scott & Zhang (1990), Clément (1975) or Oswald (1993) interpolants. Some of these interpolation operators are computed by local averaging on macro-elements, and for them our method may also be viewed as a local projection–stabilization procedure, with a similar computational cost. But we can also use any stable interpolation operator that acts on locally continuous functions and in this case only requires a single value per node. Thus, these methods have a more compact stencil while retaining the same optimal accuracy as all projection–stabilization methods, in the sense that their convergence order is optimal with respect to the degree of the finite element spaces. Furthermore, specific stabilizations for convection and pressure may be used via different quasi-interpolation operators and stabilization coefficients.

In this work, we perform the numerical analysis of our method and present some numerical tests. Stability and optimal error estimates for smooth solutions are established on the basis of a specific inf–sup condition, which is the main technical contribution of this paper. The derivation of this condition faces the difficulty of a reduced number of degrees of freedom of the buffer space. We derive two versions of this inf–sup condition: the first one holds on shape-regular meshes and the second one on quasi-uniform meshes. To circumvent quasi-uniformity, we argue locally by decomposing the mesh into quite general macro-elements; this decomposition is only used in the proof, not in the scheme. The macro-elements allow one to bound the highest frequency component of the pressure gradient. Of course, the proof simplifies when the mesh is quasi-uniform. We also include some numerical tests for 2D smooth Oseen flows that agree fairly well with the theoretical expectations of our analysis.

The paper is structured as follows. In Section 2, we introduce our numerical scheme for the Oseen equations. In Section 3, we do its numerical analysis (stability, convergence and error estimates). The short Section 4 analyses an example of interpolant that uses exclusively nodal values, and the short Section 5 considers the simpler case of uniformly regular grids. Finally, Section 6 presents some relevant numerical tests that confirm our theoretical predictions.

2. Discretization of Oseen problem

Let $\Omega$ be a bounded domain of $\mathbb{R}^d$ ($d = 2$ or $3$), with Lipschitz-continuous boundary $\Gamma$. We propose to numerically solve the following boundary value problem for the Oseen equations:

\[\begin{align*}
\text{Find } y : \tilde{\Omega} &\mapsto \mathbb{R}^d,\ p : \Omega &\mapsto \mathbb{R} \quad \text{such that,} \\
u \cdot \nabla y - \nu \Delta y + \nabla p &= f, \ \nabla \cdot y = 0 \quad \text{in } \Omega, \\
y &= 0 \quad \text{on } \Gamma.
\end{align*}\] (1)

Here, $\nu > 0$ is the viscosity coefficient, $u$ is a given velocity field such that $u \in L'(\Omega)^d$, with $r > d$ and $\nabla \cdot u = 0$, and $f \in H^{-1}(\Omega)^d$ is a given source term. To reduce nonessential difficulties, we restrict the derivation to the case of homogeneous Dirichlet boundary conditions.

We shall use the following notation for the norms either for scalar-, vector- or tensor-valued functions:
For \( \ell \neq 2 \), \( \| \cdot \|_{0,\ell} \) and \( \| \cdot \|_{0,\ell,O} \) are the norms on \( L^\ell(\Omega) \) and \( L^{\ell}(O) \), respectively, for some open subset \( O \) of \( \Omega \).

For \( \ell = 2 \), \( \| \cdot \|_0 \) and \( \| \cdot \|_{0,O} \) are the norms on \( L^2(\Omega) \) and \( L^{2}(O) \), respectively.

\( \| \cdot \|_{\ell,q} \) and \( \| \cdot \|_{\ell,q,O} \) are the norms on \( W^{\ell,q}(\Omega) \) and \( W^{\ell,q}(O) \), respectively.

\( | \cdot |_\ell \) and \( | \cdot |_{\ell,O} \) are the seminorms on \( H^\ell(\Omega) \) and \( H^{\ell}(O) \), respectively.

\( \| \cdot \|_{-1} \) and \( \| \cdot \|_{-1,O} \) are the norms on \( H^{-1}(\Omega) \) and \( H^{-1}(O) \), respectively.

We define the bilinear form on \( H^1_0(\Omega)^d \times H^1_0(\Omega)^d \) by

\[
\forall v, w \in H^1_0(\Omega)^d, \quad a(v, w) = (u \cdot \nabla v, w) + v(\nabla v, \nabla w),
\]

where (\( , , \)) and (\( , , \)) are the \( L' - L' \) duality pairings, \( 1/r + 1/r' = 1 \), in \( \Omega \) and \( O \), respectively. The form \( a( , , ) \) is well defined and continuous; i.e., it satisfies:

There exists a constant \( C \) only depending on the domain \( \Omega \) such that

\[
\forall v, w \in H^1_0(\Omega)^d, \quad a(v, w) \leq M |v|_1 |w|_1, \quad \text{with} \quad M = C(v + \|u\|_{0,r}).
\]

Also, \( a( , , ) \) is \( H^1_0(\Omega)^d \)-elliptic:

\[
\forall v \in H^1_0(\Omega)^d, \quad a(v, v) \geq \nu |v|_1^2.
\]

Problem (1) can be set in a mixed variational form as follows:

\[
\left\{ \begin{array}{l}
\text{Find} \ (y, p) \in H^1_0(\Omega)^d \times L^2_0(\Omega) \ \text{such that}, \\
L(y, p; v, q) = (F, (v, q)), \quad \forall (v, q) \in H^1_0(\Omega)^d \times L^2_0(\Omega),
\end{array} \right.
\]

where

\[
L(y, p; v, q) = a(y, v) - (p, \nabla \cdot v) + (\nabla \cdot y, q) \quad \text{and} \quad (F, (v, q)) = (f, v),
\]

\( \langle , , \rangle \) and \( \langle , , \rangle \) stand for the duality between a Hilbert space and its dual on \( \Omega \) and \( O \), respectively, and \( L^2_0(\Omega) \) is the subspace of \( L^2(\Omega) \) given by

\[
L^2_0(\Omega) = \left\{ q \in L^2(\Omega) \ \text{such that} \ \int_\Omega q \, dx = 0 \right\}.
\]

The pair of spaces \( (H^1_0(\Omega)^d, L^2_0(\Omega)) \) verifies a continuous inf–sup condition (cf., for instance, Girault & Raviart, 1986). Owing to properties (3), (4) and this inf–sup condition, Problem (5) has a unique solution that depends continuously on the data \( f \).

In order to describe our discretization of Problem (5), we assume from now on that \( \Omega \) is a polygon if \( d = 2 \) or a Lipschitz polyhedron if \( d = 3 \). Let \( \{T_h\}_{h>0} \) be a family of conforming (i.e., without hanging nodes) triangulations of \( \hat{\Omega} \) formed by simplicial elements. The parameter \( h \) is the largest diameter of the elements of \( T_h \).

Given an integer \( \ell \geq 0 \), and an element \( K \in T_h \), denote by \( \mathbb{P}_\ell(K) \) the space of polynomials of degree less than, or equal to \( \ell \), defined on \( K \). For \( \ell \geq 2 \) and \( m \geq 2 \), we consider the following finite elements
spaces:
\[
\begin{align*}
V_h^l(\Omega) &= \{ r \in C^0(\Omega) \text{ such that } r|_K \in \mathbb{P}_l(K), \forall K \in \mathcal{T}_h \}, \\
Y_h &= (V_h^l(\Omega) \cap H^1_0(\Omega))^d, \quad M_h = V_h^m \cap L^2_0(\Omega),
\end{align*}
\]

and we define the discrete dual norm of a functional \( f \) on \( Y_h \) by
\[
\| f \|_{*,h} = \sup_{v_h \in Y_h} \frac{\langle f, v_h \rangle}{|v_h|_1}.
\]

With the above spaces, we discretize problem (5) as follows:
\[
\begin{align*}
\text{Find} & \quad (y_h, p_h) \in Y_h \times M_h \quad \text{such that}, \\
L_h(y_h, p_h; v_h, q_h) &= \langle F, (v_h, q_h) \rangle, \quad \forall (v_h, q_h) \in Y_h \times M_h,
\end{align*}
\]

where
\[
L_h(y, p; v, q) = L(y, p; v, q) + \sum_{K \in \mathcal{T}_h} \tau_{pK}(\Pi_h^*(\nabla p), \Pi_h^*(\nabla q))_K + \sum_{K \in \mathcal{T}_h} \tau_{vK}(\Pi_h^*(u \cdot \nabla y), \Pi_h^*(u \cdot \nabla v))_K.
\]

Here the following conditions are satisfied:

- \( \Pi_h^* = \text{Id} - \Pi_h \), where \( \Pi_h \) is some stable approximation operator from \( L^2(\Omega)^d \) into \( [V_h^{l-1}]^d \) (called ‘buffer space’).
- \( \tau_{pK} \) and \( \tau_{vK} \) are stabilization coefficients such that there exist positive constants \( \alpha_1 \) and \( \alpha_2 \), independent of \( h \), such that for all \( K \in \mathcal{T}_h \)
  \[
  \alpha_1 h_K^2 \leq \tau_{pK} \leq \alpha_2 h_K^2 \quad \text{and} \quad \alpha_1 h_K^{2+d/r} \leq \tau_{vK} \leq \alpha_2 h_K^2.
  \]

In Section 2.1, we describe several commonly used choices of stabilization coefficients that verify estimates (8).

In the definition of \( L_h \), the extra terms added to \( L \) are least-squares terms, designed to minimize a discrete \( H^{-1} \) norm of the pressure and convection terms. They enhance the coerciveness of the bilinear form \( L_h \). Method (7) generalizes the term-by-term stabilized method introduced by Chacón Rebollo (1998), where \( \Pi_h^* = \text{Id} \), i.e., \( \Pi_h = 0 \). It differs from the local projection-based stabilization methods described in Braack et al. (2007, Section 5), because (7) uses the same mesh and its approximation operator acts on the convection or the gradient terms and not on the function itself. It differs from the local projection stabilizations described in Matthies et al. (2007) and Knobloch (2010), because (7) is defined on standard polynomial function spaces without enriching bubbles and again (7) uses a single mesh.

The stabilization term for the pressure is similar to the projection–stabilization introduced by Blasco and Codina (cf. 2000). But there are two differences with (Blasco & Codina, 2000). On the one hand, the buffer space here consists of piece-wise polynomial functions with one degree less than the finite element space for velocities, whereas they have the same degree in Blasco & Codina (2000). On the other hand, we consider here general approximation operators, and not necessarily the \( L^2(\Omega) \) projection operator.
Also, when $\Pi_h$ is the Oswald interpolation operator, method (7) essentially coincides with the interior penalty method of Braack & Burman (2000) with a specific choice of the stabilizing coefficients and up to some minor technical modifications.

We define the scalar products

$$(\cdot, \cdot)_{\tau_p} : L^2(\Omega) \times L^2(\Omega) \mapsto \mathbb{R},$$

$$(f, g) \mapsto \sum_{K \in T_h} \tau_p(f, g)_K,$$

and their associated norms

$$\|f\|_{\tau_p} = (f, f)_{\tau_p}^{1/2}, \quad \|f\|_{\tau_v} = (f, f)_{\tau_v}^{1/2}. \quad (9)$$

The following well-known local inverse inequalities will be frequently used in the sequel. Their proof is a standard application of norm equivalence on finite-dimensional spaces.

**Lemma 2.1** For each integer $\ell \geq 1$, there exists a constant $\hat{C}$ independent of $h$ such that

$$\forall K \in T_h, \forall p \in P_\ell(K), \quad |p|_{1,K} \leq \hat{C} \frac{1}{\rho_K} \|p\|_{0,K}. \quad (10)$$

Similarly, for each real number $r \geq 1$, there exists a constant $\hat{C}$ independent of $h$ such that

$$\forall K \in T_h, \forall p \in P_\ell(K), \quad \|p\|_{0,r,K} \leq \hat{C} \frac{h_K^{d/r}}{\rho_K} \|p\|_{0,K}. \quad (11)$$

In all this work, we make the following assumptions on $T_h$.

**Hypothesis 2.2**

1. The family $\{T_h\}_{h>0}$ is shape-regular in the sense of Ciarlet (1978): there exists a constant $\sigma > 0$ independent of $h$ such that

$$\forall K \in T_h, \frac{h_K}{\rho_K} \leq \sigma, \quad (12)$$

where $h_K$ is the diameter of $K$ and $\rho_K$ is the diameter of the ball inscribed in $K$.

2. No element of $T_h$ has 3 nodes (when $d = 2$) or 4 nodes (when $d = 3$) on the boundary of $\Omega$.

In Section 3, in order to establish a suitable inf–sup condition without assuming that the mesh is quasi-uniform, we decompose $\bar{\Omega}$ into a finite union of macro-elements $\mathcal{O}_i$ such that:

1. $\bar{\Omega} = \bigcup_{i=1}^{R} \mathcal{O}_i$;

2. Any element $K \in T_h$ can belong to at most $M$ macro-elements, where $M$ is independent of $h$. 
(3) Each macro-element \( O_i \) contains the support of at least one piece-wise affine basic function.

The existence of such a decomposition is guaranteed by Hypothesis 2.2: it is enough to choose for \( O_i \) the support of the \( P_1 \) basis functions.

Let \( V_h^L(O_i) \) and \( Y_h(O_i) \) denote the analogues of \( V_h^L(\Omega) \) and \( Y_h \) defined on each \( O_i \):

\[
\begin{align*}
\left\{ \begin{array}{l}
V_h^L(O_i) = \{ r \in C^0(\overline{O_i}) \text{ such that } r_k \in P_1(K), \forall K \subset O_i \}, \\
Y_h(O_i) = (V_h^L(O_i) \cap H^1_0(O_i))^d.
\end{array} \right.
\end{align*}
\]

(13)

Next, we make the following local assumptions on \( \Pi_h \). For any \( K \) in \( T_h \), we denote by \( \omega_K \) the union of all elements of \( T_h \) that intersect \( K \). We shall denote throughout the paper by \( C, C_1, C_2, \ldots \) constants that may vary from a line to another, but which always are independent of \( h \).

**Hypothesis 2.3** There exists a constant \( C \) independent of \( h \) such that

\[
\forall g \in L^2(\Omega)^d, \forall K \in T_h, \quad \| \Pi_h(g) \|_{0,K} \leq C \| g \|_{0,\omega_K}.
\]

(14)

In other words, \( \Pi_h \) is locally stable in \( L^2(\Omega)^d \). Owing to the local quasi-uniformity of regular meshes, Hypothesis 2.2 and (14) imply that \( \Pi_h \) is globally stable in \( L^2(\Omega)^d \).

**Hypothesis 2.4** The operator \( \Pi_h \) satisfies the following local approximation estimates (recall \( \ell \geq 2 \)):

\[
\forall v \in W^{\ell,p}(\Omega)^d, \forall K \in T_h, \quad |v - \Pi_h(v)|_{r,p,K} \leq C_p h_k^{\ell-r} |v|_{\ell,p,\omega_K}, \quad r = 0, 1, 1 \leq p \leq +\infty, \ r \leq \ell.
\]

(15)

These two assumptions are verified by quasi-local approximation operators such as the Girault–Lions, Bernardi–Maday–Rapetti, Clément, Oswald or Scott–Zhang-type operators and local \( L^2 \) interpolation operators such as the Ganesan–Matthies–Tobiska operator, but not by the global \( L^2(\Omega) \) projection. However, the Ganesan–Matthies–Tobiska operator cannot be used in this context because it takes its values in a discontinuous finite element space.

Finally, we assume some additional smoothness on the data \( f \).

**Hypothesis 2.5** There exists a constant \( C \), independent of \( h \), such that

\[
\sum_{i=1}^R \| f \|_{-1,O_i}^2 \leq C.
\]

(16)

This condition holds for instance if \( f \in L^2(\Omega)^d \).

Under these assumptions, method (7) is stable and has optimal convergence rate for smooth solutions. This is stated below in Theorems 3.8 and 3.9. Of course, when the mesh is quasi-uniform, the proofs are simpler, assumption (16) is not necessary, assumptions 2.3 and 2.4 can be stated globally, and we reach similar conclusions. This case is briefly discussed in Section 5.

**Remark 2.6** In this work, as we are interested in the case when \( \nu \) is small: the independence of all constants on \( \nu \) is to be understood for \( \nu \) in the range \( ]0, \nu_0[ \) for a fixed \( \nu_0 > 0 \).
2.1 Stabilization coefficients

Estimates (8) are satisfied by various stabilization coefficients used in practice. For instance, a choice for the convection stabilization coefficients, that takes into account convection and diffusion-dominated regimes is (cf. Chacón Rebollo, 1998; Franca & Frey, 1992; Knobloch, 2010):

\[
\tau_{vK} = A \frac{h_K}{U_K} \min\{\text{Pe}_K, P\} = \begin{cases} A \frac{h_K^2}{v} & \text{if } \text{Pe}_K \leq P, \\ AP \frac{h_K}{U_K} & \text{if } \text{Pe}_K > P, \end{cases} \tag{17}
\]

where

\[
U_K = \begin{cases} \|u\|_{0,r,K} & \text{if } 1 \leq r < +\infty, \\ |K|^{1/r} & \text{if } r = +\infty, \\ \|u\|_{0,\infty,K} & \text{if } r = +\infty, \end{cases} \tag{18}
\]

Here, \(\text{Pe}_K\) is the element Péclet number, \(A > 0\) is a numerical constant, \(P > 0\) is the threshold between the diffusion- and convection-dominated regimes, and \(|K|\) denotes the measure of the set \(K\). Constants \(A\) and \(P\) depend on the actual finite element space for velocities, in such a way that \(\tau_{vK}\) decreases as the degree of the interpolation polynomial increases. Standard estimates prove that if the family of triangulations \(T_h\) is regular (Hypothesis 2.2), (17) implies that

\[
\alpha_1(u) h_K^{2+d/r} \leq \tau_{vK} \leq \alpha_2 h_K^2, \quad \text{with } \alpha_1(u) = A \min\left\{ \frac{1}{v}, \frac{P\tilde{\sigma}^{1/r}}{\text{diam}(\Omega) \|u\|_{0,r,\Omega}} \right\}, \quad \alpha_2 = \frac{A}{v},
\]

where \(\tilde{\sigma}\) (independent of \(h\)) is related to the constant \(\sigma\) of (12) by

\[
\frac{h_K^d}{|K|} \leq \tilde{\sigma}.
\]

The strongest stabilization of convection is achieved when \(r = +\infty\), for which the \(\tau_{vK}\) are of order \(h_K^2\). However, we have preferred to consider the more general case when \(u \in L^r(\Omega)^d\), considering that in practice \(u\) solves the Navier–Stokes, or similar, equations, and then its natural regularity is \(u \in H^1(\Omega)^d\), which is not injected in \(L^\infty(\Omega)^d\) for \(d \geq 2\).

Next, Codina (2000) proposes the following expression for \(\tau_{vK}\):

\[
\tau_{vK} = \left[ \left( C_1 \frac{v}{h_K^2} \right)^2 + \left( C_2 \frac{U_K}{h_K} \right)^2 \right]^{-1/2}. \tag{19}
\]

Then

\[
\lim_{U_K \to 0} \tau_{vK} = \frac{1}{C_1 v}, \quad \lim_{v \to 0} \tau_{vK} = \frac{1}{C_2 U_K},
\]

and consequently this choice asymptotically recovers (17) with \(A = 1/C_1\), \(P = C_1/C_2\). Furthermore, the SUPG solution of the one-dimensional convection–diffusion equation with constant velocity \(U\) and diffusion \(v\), with continuous piece-wise linear finite elements on a uniform grid, is nodally exact if (cf.
\[ \tau_{vK} = \frac{h}{2|U|} \xi(Pe_h), \quad \text{with} \quad \xi(\alpha) = \coth \alpha - \frac{1}{\alpha}, \quad Pe_h = \frac{|U|h}{2\nu}. \]

This leads to the same asymptotic behaviour:

\[ \lim_{U \to 0} \tau_{vK} = \frac{1}{12} \frac{h^2}{\nu}, \quad \lim_{\nu \to 0} \tau_{vK} = \frac{1}{2} \frac{h}{|U|}. \]

Finally, static condensation of the bubbles for the mini-element yield typical pressure stabilization coefficients (cf. Chacón Rebollo, 1998; Franca & Frey, 1992),

\[ \tau_{pK} = \frac{C_d}{v} \frac{|K|}{|b_K|^2}, \]

where \( C_d \) is a numerical constant that depends on the dimension \( d \) and \( b_K \) is the bubble function on \( K \) (the product of the barycentric coordinates on \( K \)). If the grids are regular, this choice of \( \tau_{pK} \) satisfies (8). A simpler choice having the same order with respect to \( h_K \) is

\[ \tau_{pK} = \frac{C_3}{v} h^2, \quad C_3 > 0. \quad (20) \]

3. The case of shape-regular meshes

Throughout this section we assume that Hypothesis 2.2 holds and that \( \Omega \) is split into macro-elements as in Section 2. Let

\[ h_i = \max\{h_K, \ K \subset \mathcal{O}_i\}, \quad \rho_i = \min\{\rho_K, \ K \subset \mathcal{O}_i\}. \]

Part 1 of Hypothesis 2.2 implies that the mesh is locally quasi-uniform and in turn this implies that there exist positive constants \( C_1, C_2 \) and \( C_3 \) independent of \( h \), such that, for all \( K \in \mathcal{T}_h \) and for all \( i \) for which \( K \subset \mathcal{O}_i \),

\[ C_1 h_i \leq h_K \leq C_2 h_i, \quad \frac{h_i}{\rho_i} \leq C_3. \quad (21) \]

We have the following properties of \( \Pi^*_h \).

**Lemma 3.1** Let Hypothesis 2.2 hold. If the stabilization coefficients satisfy (8), then the following conditions are satisfied:

1. There exist positive constants \( C_1 \) and \( C_2 \), independent of \( h \) such that

\[ \forall z \in L^2(\Omega), \quad C_1 \sum_{i=1}^{R} h_i^2 \|z\|^2_{0,\mathcal{O}_i} \leq \|z\|^2_{\tau} \leq C_2 \sum_{i=1}^{R} h_i^2 \|z\|^2_{0,\mathcal{O}_i}, \quad (22) \]

where \( \tau \) denotes either \( \tau_v \) or \( \tau_p \).

2. If in addition Hypothesis 2.3 holds, there exists a constant \( C_3 > 0 \) independent of \( h \), such that

\[ \forall g \in L^2(\Omega), \quad \|\Pi^*_h(g)\|_{\tau} \leq C_3 h\|g\|_0. \quad (23) \]
(3) Under the same assumptions, there exists a constant $C_4 > 0$ independent of $h$, such that
\[ \forall f \in V_h^\ell, \quad \| \Pi_h^*(\nabla f) \|_\tau \leq C_4 \| f \|_0. \] (24)

\textbf{Proof.} \hspace{1cm} (1) Let $z \in L^2(\Omega)$. Denote by $M$ the maximum number of macro-elements that contain any element $K \in T_h$ and recall that owing to Hypothesis 2.2, $M$ is independent of $h$. Then, in view of (8),
\[ M \| z \|_T^2 \geq \sum_{i=1}^R \sum_{K \subset O_i} \tau_K \| z \|_{0,K}^2 \geq C \sum_{i=1}^R \sum_{K \subset O_i} h_K^2 \| z \|_{0,K}^2. \]
By (21), and as $\sum_{K \subset O_i} \| z \|_{0,K}^2 = \| z \|_{0,O_i}^2$, we obtain
\[ \| z \|_T^2 \geq C \sum_{i=1}^R h_i^2 \| z \|_{0,O_i}^2. \]
The other estimate in (22) follows from (8), (21), and the properties of the macro-elements.

(2) Let $g \in L^2(\Omega)$. By applying the second part of (22) to $\Pi_h^*(g)$, we obtain
\[ \| \Pi_h^*(g) \|_\tau^2 \leq C_2 \sum_{i=1}^R h_i^2 \| \Pi_h^*(g) \|_{0,O_i}^2 \leq C_2 \sum_{i=1}^R \| \Pi_h^*(g) \|_{0,O_i}^2. \]
Now, Hypothesis 2.3 implies that
\[ \| \Pi_h^*(g) \|_{0,O_i}^2 \leq C \sum_{K \subset O_i} \| g \|_{0,o_K}^2. \]
Therefore, summing over all $i$ and using again the fact that the mesh is locally quasi-uniform, we conclude
\[ \| \Pi_h^*(g) \|_\tau \leq C_3 h \| g \|_0. \]

(3) Let $f \in V_h^\ell$. It stems from (8), (21) and Hypothesis 2.3 that
\[ \| \Pi_h^*(\nabla f) \|_\tau^2 \leq C \sum_{K \subset O_i} h_K^2 \| \Pi_h^*(\nabla f) \|_{0,K}^2 \leq C \sum_{K \subset O_i} h_K^2 \| \nabla f \|_{0,o_K}^2. \]
The local inverse inequality (10) gives
\[ \| \nabla f \|_{0,o_K}^2 \leq C \sum_{T \subset o_K} \frac{1}{\rho_T^2} \| f \|_{0,T}^2 \leq \max_{T \subset o_K} \frac{C}{\rho_T^2} \| f \|_{0,o_K}^2. \] (25)
But again, the local quasi-uniformity of the mesh implies that there exists a constant $C$, independent of $h$, such that for all $i$ and for all $K \in O_i$,
\[ h_i \leq C \min_{T \subset o_K} \rho_T. \]
Therefore
\[ \| \Pi_h^e(\nabla f) \|_T^2 \leq C \sum_{i=1}^R \sum_{K \subset O_i} \| f \|_{0,\partial K}^2 \leq C \| f \|_0^2. \]

This yields (24). \qed

3.1 Inf–sup condition

In this section, we prove an inf–sup condition associated with method (7) that is essential for its stability. The main difficulty in this proof stems from the facts that on the one hand, the mesh is not quasi-uniform and on the other hand, the interpolation operator \( \Pi_h \) takes values in \( V_h^{\ell-1} \), thus reducing the effective number of degrees of freedom of the velocity space.

We start with the following auxiliary result stated in \( \Omega \); then we shall apply it with \( O_i \) instead of \( \Omega \). Its statement in \( O_i \) will be analogous to that of Knobloch (2010, Lemma 5.1), except that the statement of Lemma 5.1 only applies to functions in \( P^{\ell-1}_h(O_i) \), whereas here we shall consider all functions in \( V_h^{\ell-1}(O_i) \).

**Lemma 3.2** If Hypothesis 2.2 holds, there exists a constant \( C > 0 \) independent of \( h \), such that
\[
\forall g_h \in V_h^{\ell-1}, \quad \| g_h \|_0 \leq C \sup_{v_h \in V_h^{\ell} \cap H_0^1(\Omega)} \frac{(v_h, g_h)}{\| v_h \|_0}. \tag{26}
\]

**Proof.** Part of the proof can be found in Knobloch (2010, Lemma 5.1), but we reproduce it for the readers’ convenience. Let \( \{a_i\} \) denote the nodes of the triangulation \( T_h \) and define \( w_h \in V_h^1 \cap H_0^1(\Omega) \) by
\[
w_h|_K \in P^1(K) \text{ for all } K \in T_h, \\
w_h(a_i) = 1, \text{ if } a_i \text{ is an interior node}, \\
w_h(a_i) = 0, \text{ if } a_i \text{ is a node belonging to } \Gamma.
\]

Let \( g_h \) be an arbitrary function in \( V_h^{\ell-1} \) and define \( \tilde{v}_h = g_h w_h \); then \( \tilde{v}_h \in V_h^{\ell} \cap H_0^1(\Omega) \).

1. Assume for the moment that there exists a constant \( C > 0 \) independent of \( h \) such that
\[
|\langle \tilde{v}_h, g_h \rangle| \geq C \| g_h \|_0^2. \tag{27}
\]
As
\[
\| \tilde{v}_h \|_0 = \| g_h w_h \|_0 \leq \| w_h \|_\infty \| g_h \|_0 \leq \| g_h \|_0, \tag{28}
\]
then (27) and (28) imply
\[
\frac{|\langle \tilde{v}_h, g_h \rangle|}{\| \tilde{v}_h \|_0} \geq \frac{|\langle \tilde{v}_h, g_h \rangle|}{\| g_h \|_0} \geq C \| g_h \|_0.
\]

Consequently,
\[
\forall g_h \in V_h^{\ell-1}, \quad \| g_h \|_0 \leq C^{-1} \sup_{v_h \in V_h^{\ell} \cap H_0^1(\Omega)} \frac{(v_h, g_h)}{\| v_h \|_0}. \]
(2) Now, we prove the lower bound in (27). The function \( w_h \) is continuous and owing to part 2 of Hypothesis 2.2, it is positive in \( \Omega \), thus the bilinear form

\[
\forall f, g \in V_h^{\ell - 1}, \quad (f, g)_{w_h} = \int_{\Omega} fgw_h,
\]

is an inner product in \( V_h^{\ell - 1} \). Let \( \| \cdot \|_{w_h} \) be its associated norm. As all norms are equivalent on the finite-dimensional space \( V_h^{\ell - 1} \), the norm \( \| \cdot \|_{w_h} \) is equivalent to the \( L^2(\Omega) \) norm. Therefore, there exists a constant \( C_h > 0 \) such that

\[
\forall g_h \in V_h^{\ell - 1}, \quad \|g_h\|_0 \leq C_h \|g_h\|_{w_h}.
\]

(3) The remaining part of the proof is devoted to showing that the constant \( C_h \) is independent of \( h \). With obvious notation, we write

\[
\|g_h\|_{w_h,K}^2 = \int_{\Omega} |g_h|^2 w_h = \sum_{K \in T_h} \int_K |g_h|^2 w_h = \sum_{K \in T_h} \|g_h\|_{w_h,K}^2. \tag{29}
\]

Given \( K \in T_h \), let \( g_h|_{K} = \hat{g}_h \circ (F_K)^{-1} \) and \( w_h|_{K} = \hat{w}_h \circ (F_K)^{-1} \), where \( F_K \) is the affine mapping that transforms \( \hat{K} \) onto \( K \), i.e., \( F_K(\hat{x}) = A_K \hat{x} + b_K \), for some nonsingular matrix \( A_K \) of \( \mathbb{R}^{d \times d} \) and some vector \( b_K \in \mathbb{R}^d \). Then, \( \hat{g}_h \in \mathbb{P}_{\ell - 1}(\hat{K}), \hat{w}_h \in \mathbb{P}_1(\hat{K}) \) and

\[
\|g_h\|_{w_h,K}^2 = \int_{\hat{K}} |\hat{g}_h|^2 \hat{w}_h = |\det A_K| \int_{\hat{K}} |\hat{g}_h|^2 \hat{w}_h = |\det A_K| \|\hat{g}_h\|_{\hat{w}_h}^2.
\]

Note that \( \hat{w}_h|_{\hat{K}} \) is positive in the interior of \( \hat{K} \) because, by assumption, \( K \) has at most \( d \) vertices on \( \Gamma \). Therefore, the mapping \( \hat{g}_h \mapsto \|\hat{g}_h\|_{\hat{w}_h} \) defines a norm on \( \mathbb{P}_{\ell - 1}(\hat{K}) \). By the equivalence of norms in \( \mathbb{P}_{\ell - 1}(\hat{K}) \), there exists a constant \( \hat{C}_K > 0 \) such that

\[
\|\hat{g}_h\|_{\hat{w}_h} \geq \hat{C}_K \|\hat{g}_h\|_{0,\hat{K}}.
\]

The constant \( \hat{C}_K \) is independent of \( K \) because \( \hat{w}_h \) belongs to the finite set of affine functions defined on \( \hat{K} \) that vanish at either 0, 1, \ldots, or \( d \) vertices and take the value 1 at the other nodes. Thus, the constant \( \hat{C}_K \) can take at most a fixed number of values independently of \( h \) and \( K \). Therefore there exists a constant \( \hat{C} > 0 \) independent of \( h \) such that

\[
\|\hat{g}_h\|_{\hat{w}_h} \geq \hat{C} \|\hat{g}_h\|_{0,\hat{K}}.
\]

Then

\[
\|g_h\|_{w_h,K}^2 \geq |\det A_K| \|\hat{g}_h\|_{0,\hat{K}}^2 \geq \hat{C}^2 \|g_h\|_{0,K}^2.
\]

Reverting to (29), we conclude that

\[
\|g_h\|_0 \leq C \|g_h\|_{w_h},
\]

with a constant \( C \) independent of \( h \) and \( g_h \).

Note that the difficulty in deriving (26) is that the elements of \( V_h^{\ell} \cap H^1_0(\Omega) \) vanish on \( \partial \Omega \). If we allow \( v_h \) to vary in the whole space \( V_h^{\ell} \), condition (26) is immediate.
Remark 3.3 Lemma 3.2 still holds true for Quadrilateral or Hexahedral finite elements provided that the family of triangulations is regular, i.e., the elements are strictly convex and the Jacobian $J_K$ of the bilinear transformation $F_K$ from the unit element $\hat{K}$ onto the element $K$ satisfies

$$\forall \hat{x} \in \hat{K}, \quad ch^d_K \leq J_K(\hat{x}) \leq Ch^d_K,$$

with constants $c$ and $C$ independent of $K$ and $h$. The reason is that, on one hand, in each element the function $w_h$ is a linear combination with non-negative coefficients of the inverse image of basis functions of $\hat{Q}_1$ and hence is positive inside $\Omega$. As usual, $\hat{Q}_\ell$ denotes the space of polynomials of degree $\ell$ in each variable. On the other hand, if $g_h$ belongs to the inverse image of $\hat{Q}_{\ell - 1}$, then $w_h g_h$ belongs to the inverse image of $\hat{Q}_{\ell}$. Moreover, the mapping $\hat{g}_h \mapsto \| \hat{g}_h \|_{w_h}$ in $\hat{K}$ defines an equivalent norm on $\hat{Q}_{\ell - 1}(\hat{K})$ with equivalence constants independent of $K$. Then the steps of the proof of Lemma 3.2 extend to this case and yield

$$\forall K \in T_h \quad \| g_h \|^2_{w_h,K} \geq \hat{c}^2 \frac{\max_{\hat{x} \in \hat{K}} J_K(\hat{x})}{\min_{\hat{x} \in \hat{K}} J_K(\hat{x})} \| g_h \|^2_{0,K},$$

with the same notation.

By applying the proof of Lemma 3.2 to $O_i$ and observing that if $O_i$ contains the support of one piece-wise affine basis function, then the function $w_h$ is positive in the interior of $O_i$, we immediately derive the following result.

Lemma 3.4 Let $\Omega$ be decomposed into macro-elements as in Section 2. Under Hypothesis 2.2, there exists a constant $C > 0$ independent of $h$ such that, for all $i, 1 \leq i \leq R$,

$$\forall g_h \in V_{h}^{\ell - 1}(O_i), \quad \| g_h \|_{0,O_i} \leq C \sup_{v_h \in V_h^d(O_i) \cap H_0^1(O_i)} (v_h, g_h)_{O_i}.$$  \hfill (30)

Note that the statement of Lemma 3.4 is not true when the functions of $V_{h}^{\ell - 1}(O_i)$ are discontinuous at element interfaces. An analogous remark applies to Lemma 3.2.

Then we have the following inf–sup condition. Note that its statement requires no particular stability property of $\Pi_h$.

Lemma 3.5 Let $\Omega$ be decomposed into macro-elements as in Section 2. Under Hypothesis 2.2, the following inf–sup condition holds: there exists a constant $C > 0$ independent of $h$ such that, for all $q_h \in M_h$,

$$C \| q_h \|_0 \leq \sup_{v_h \in Y_h} \frac{(\nabla \cdot v_h, q_h)}{|v_h|_1} + \| \Pi_h^*(\nabla q_h) \|_{\tau_p} + \left( \sum_{i=1}^{R} \left( \sup_{w_h \in Y_h(O_i)} \frac{(\nabla \cdot w_h, q_h)_{O_i}}{|w_h|_1, O_i} \right)^2 \right)^{1/2},$$

where the space $Y_h(O_i)$ is defined in (13).

Proof. Here the various constants $C$ are independent of $h$.

(1) Let us recall briefly the proof of the following weak inf–sup condition valid on a shape-regular mesh (cf. Franca et al., 1993; Verfürth, 1984): There exist two positive constants $C_1$ and $C_2$ independent
of \( h \) such that
\[
\forall q_h \in M_h, \quad C_1 \| q_h \|_0 \leq \sup_{\nu_h \in Y_h} \frac{(\nabla \cdot \nu_h, q_h)}{\| \nu_h \|_1} + C_2 \| \nabla q_h \|_{r_p}.
\]

Indeed, by virtue of the continuous inf–sup condition, for each \( q_h \in M_h \), there exists \( \nu \in H_0^1(\Omega)^d \) and a constant \( C \) independent of \( q_h \) and \( \nu \) such that
\[
\nabla \cdot \nu = q_h \quad \text{in} \quad \Omega, \quad |\nu|_1 \leq C \| q_h \|_0.
\]

Then applying Green’s formula, we have for any \( \nu_h \in Y_h \),
\[
\| q_h \|_0^2 = (\nabla \cdot \nu_h, q_h) + (\nabla \cdot (\nu - \nu_h), q_h) = (\nabla \cdot \nu_h, q_h) - (\nu - \nu_h, \nabla q_h).
\]

Take for \( \nu_h \) a suitable interpolant of \( \nu \) satisfying in each \( K \)
\[
\| \nu - \nu_h \|_{0,K} \leq C h_K |\nu|_{1,\omega_K},
\]
with a constant \( C \) independent of \( K \) and \( h \). Then (8) and the local quasi-uniformity of the meshes imply
\[
|(\nu - \nu_h, \nabla q_h)| \leq C \sum_{K \in T_h} h_K |q_h|_{1,K} |\nu|_{1,\omega_K} \leq C \| \nabla q_h \|_{r_p} |\nu|_1,
\]
whence (32).

(2) In order to bound the term \( \| \nabla q_h \|_{r_p} \), we use the fact that \( \Pi_h + \Pi_h^* = I_d \) and we write
\[
\| \nabla q_h \|_{r_p} \leq \| \Pi_h(\nabla q_h) \|_{r_p} + \| \Pi_h^*(\nabla q_h) \|_{r_p}.
\]

Inequality (22) gives
\[
\| \Pi_h(\nabla q_h) \|_{r_p}^2 \leq C \sum_{i=1}^R h_i^2 \| \Pi_h(\nabla q_h) \|_{0,\Omega_i}^2.
\]

As \( \Pi_h(\nabla q_h)|_{\Omega_i} \in V_h^{l-1}(\Omega_i)^d \), we can apply the inf–sup condition (30):
\[
\| \Pi_h(\nabla q_h) \|_{0,\Omega_i}^2 \leq C \sup_{w \in Y_h(\Omega_i)} \frac{|(\Pi_h(\nabla q_h), w)_{\Omega_i}|}{\| w \|_{0,\Omega_i}}.
\]

Using again \( \Pi_h + \Pi_h^* = I_d \), we have
\[
\| \Pi_h(\nabla q_h) \|_{0,\Omega_i}^2 \leq C \left( \sup_{w \in Y_h(\Omega_i)} \frac{|(\nabla q_h, w_h)_{\Omega_i}|}{\| w_h \|_{0,\Omega_i}} \right)^2 + C \| \Pi_h^*(\nabla q_h) \|_{0,\Omega_i}^2,
\]
and reverting to (34), we obtain
\[
\| \Pi_h(\nabla q_h) \|_{r_p}^2 \leq C \sum_{i=1}^R h_i^2 \sup_{w_h \in Y_h(\Omega_i)} \frac{|(\nabla q_h, w_h)_{\Omega_i}|^2}{\| w_h \|_{0,\Omega_i}^2} + C \sum_{i=1}^R h_i^2 \| \Pi_h^*(\nabla q_h) \|_{0,\Omega_i}^2 = I + II.
\]

The second term is estimated by the first part of inequality (22),
\[
|II| \leq C \| \Pi_h^*(\nabla q_h) \|_{r_p}^2.
\]
To estimate the first term, we use the local inverse inequality (25) in $\mathcal{O}_i$:

$$|w_h|_{1,\mathcal{O}_i} \leq \frac{C}{\rho_i} \|w_h\|_{0,\mathcal{O}_i}.$$  \hspace{1cm} (35)

Then (21) yields

$$\sum_{i=1}^R h_i^2 \sup_{w_h \in Y_h(\mathcal{O}_i)} |(\nabla q_h, w_h)_{\mathcal{O}_i}|^2 \leq C \sum_{i=1}^R h_i^2 \sup_{w_h \in Y_h(\mathcal{O}_i)} \|w_h\|_{0,\mathcal{O}_i}^2 \leq C \sum_{i=1}^R \sup_{w_h \in Y_h(\mathcal{O}_i)} |(\nabla q_h, w_h)_{\mathcal{O}_i}|^2 \leq C \sum_{i=1}^R \sup_{w_h \in Y_h(\mathcal{O}_i)} |(\nabla q_h, w_h)_{\mathcal{O}_i}|^2.$$  

Finally, since $Y_h(\mathcal{O}_i) \subset H^1_0(\mathcal{O}_i)^d$, we have by Green’s formula

$$(\nabla q_h, w_h)_{\mathcal{O}_i} = -(\nabla \cdot w_h, q_h)_{\mathcal{O}_i}.$$  \hspace{1cm} \Box

Observe that $\Pi_h^*(\nabla q_h)$ includes the high-frequency components of $\nabla q_h$ that cannot be represented in $Y_h$ by means of the interpolation operator $\Pi_h$. Therefore, in the inf–sup condition (31), the first term in the right-hand side bounds the components of $\nabla q_h$ that are representable in $Y_h$, and the last two terms bound the components that are not. In particular, the last term, which is absent from (31) when the mesh is quasi-uniform (cf. Lemma 5.4), gives some extra control on the high frequencies of $\nabla q_h$ possibly caused by the nonuniform regularity of the meshes.

**Remark 3.6** Lemma 3.5 still holds true for Quadrilateral or Hexahedral finite elements if the family of triangulations is regular (see Remark 3.3). In this case, the approximation properties and inverse inequalities of standard finite element subspaces of $H^1(\Omega)$ are the same as those built on simplices, and with Lemma 3.2, only these properties are used in the proof of Lemma 3.5.

Next, we shall deduce from Lemma 3.4 an interpolant $R_h$ of smooth functions such that: for all $y \in C^0(\Omega)$, $R_hy \in V_h^\ell$ satisfies

$$\forall q_h \in V_h^{\ell-1}, \quad (R_hy - y, q_h) = 0,$$  \hspace{1cm} (36)

with the same optimal approximation properties as the standard nodal Lagrange interpolant $I_h^\ell \in L(C^0(\Omega); V_h^\ell)$ (see, for instance, Ciarlet, 1978): For $1 \leq s \leq \ell$, $k = 0, 1$, $\forall K \in T_h$,

$$\forall y \in (H^{s+1}(\Omega) \cap H^1_0(\Omega))^d, \quad \|y-I_h^\ell y\|_{k,q,K} \leq C h_h^{-k+1/d(q-d)/2} |y|_{s+1,K}, \quad 1 \leq q \leq +\infty,$$  \hspace{1cm} (37)

with some constant $C > 0$ independent of $h$.

**Lemma 3.7** Under the assumptions of Lemma 3.4, there exists an interpolant $R_h \in L(C^0(\Omega); V_h^\ell)$ satisfying (36) and, for $1 \leq s \leq \ell$, $k = 0, 1$, $1 \leq i \leq R$,

$$\forall y \in (H^{s+1}(\Omega) \cap H^1_0(\Omega))^d, \quad \|y-R_hy\|_{k,q,\mathcal{O}_i} \leq C h_h^{-k+1/d(q-d)/2} |y|_{s+1,\mathcal{O}_i}, \quad 1 \leq q \leq +\infty,$$  \hspace{1cm} (38)

with a constant $C > 0$ independent of $h$ and $i$. 


Proof. The proof is essentially part of the proof of Theorem 1 in Girault & Scott (2003), but we include it here for the readers’ convenience. In the case when the macro-elements are not a partition of \( \Omega \), we associate with them the following partition \( \{ \Delta_i \}_{i=1}^R \) of \( \tilde{\Omega} \): \( \Delta_1 = \mathcal{O}_1 \), and for \( 2 \leq i \leq R \), \( \Delta_i \) is the set (possibly empty) of all \( K \) that belong to \( \mathcal{O}_i \) but not to \( \bigcup_{j=1}^{i-1} \Delta_j \). By construction, the \( \Delta_i \) are mutually disjoint,

\[
\tilde{\Omega} = \bigcup_{i=1}^R \Delta_i, \quad \Delta_i \subset \mathcal{O}_i, \quad 1 \leq i \leq R.
\]

Owing to (30) and the Babuška–Brezzi’s theory (see for instance Girault & Raviart, 1986), given \( y \) in \( C_0^0(\bar{\Omega}) \), there exists a function \( c_{h,i}(y) \in Y_h(\mathcal{O}_i) \), unique in an orthogonal space, such that

\[
\forall q_h \in V_h^{l-1}(\mathcal{O}_i), \quad (c_{h,i}(y), q_h)_{\mathcal{O}_i} = (y - I_h^l y, q_h)_{\Delta_i}, \quad c_{h,i}(y) = 0 \quad \text{if} \quad \Delta_i = \emptyset,
\]

\[
\|c_{h,i}(y)\|_{0,\mathcal{O}_i} \leq \frac{1}{C} \|y - I_h^l y\|_{0,\Delta_i},
\]

with the constant \( C \) of (30); recall that it is independent of \( h \) and \( i \). Let us extend \( c_{h,i} \) by zero outside \( \mathcal{O}_i \) and define

\[
R_h y = I_h^l y + c_h(y), \quad c_h(y) = \sum_{i=1}^R c_{h,i}(y).
\]

Then we infer from the support of \( c_{h,i}(y) \) and the fact that \( \{ \Delta_i \}_{i=1}^R \) is a partition of \( \tilde{\Omega} \),

\[
(c_h(y), q_h) = \sum_{i=1}^R (c_{h,i}(y), q_h) = \sum_{i=1}^R (c_{h,i}(y), q_h)_{\mathcal{O}_i} = \sum_{i=1}^R (y - I_h^l y, q_h)_{\Delta_i} = (y - I_h^l y, q_h),
\]

whence (36). Furthermore,

\[
\|y - R_h y\|_{0,\mathcal{O}_i} \leq \left(1 + \frac{1}{C}\right) \|y - I_h^l y\|_{0,\mathcal{O}_i}, \quad (40)
\]

Then the interpolation estimates (38) follow from (40), (37), repeated applications of the local inverse estimates (10) and (11), and the fact that each macro-element contains at most \( L \) elements \( K \), with \( L \) independent of \( h \). \( \square \)

When the macro-elements form a partition of \( \Omega \), \( R_h \) is similar to the special interpolation operator \( j_h \) constructed in Matthies et al. (2007).

In the case of Quadrilateral or Hexahedral finite elements, the seminorms in the right-hand sides of (37) and (38) are replaced by full norms (but without the \( L^2 \) norm).

3.2 Stability analysis

In the next theorem, we prove the stability of discretization (7), using the inf–sup condition (31).
Theorem 3.8 Under the assumptions of Lemma 3.5 and if Hypotheses 2.3 and 2.5 hold, then problem (7) has a unique solution. Moreover, there exists a constant $C > 0$, independent of $h$ and $\nu$, such that

$$\nu |y_h|_1 + \sqrt{\nu} \| \Pi_h^* (\nabla p_h) \|_{\tau_p} + \sqrt{\nu} \| \Pi_h^* (u \cdot \nabla y_h) \|_{\tau_v} \leq 2 \| f \|_{s,h}, \quad (41)$$

and

$$\| p_h \|_0 \leq C \left( 1 + \frac{1}{\sqrt{\nu}} + \frac{\| u \|_{0,0}}{\nu} \right) \| f \|_{s,h} + C \left( \sum_{i=1}^{R} \| f \|_{-1,\mathcal{O}_i}^2 \right)^{1/2}. \quad (42)$$

Proof. The various constants here are independent of $h$ and $\nu$.

Problem (7) is equivalent to a square linear system of dimension $\dim Y_h + \dim M_h$ equations. Hence uniqueness of the solution is equivalent to its existence. Let us assume that there exists a solution and prove that it is unique. Since this will follow from (41) and (42), we only need to prove these estimates for any solution of problem (7).

(1) By choosing in (7) the test functions $v_h = y_h$ and $q_h = p_h$, we have

$$a(y_h, y_h) + \sum_{K \in T_h} \tau_p K \| \Pi_h^* (\nabla p_h) \|_{0,K}^2 + \sum_{K \in T_h} \tau_v K \| \Pi_h^* (u \cdot \nabla y_h) \|_{0,K}^2 = \langle f, y_h \rangle. \quad (45)$$

The ellipticity (4) of $a$ implies first

$$|y_h|_1 \leq \frac{1}{\nu} \| f \|_{s,h}. \quad (43)$$

Next, Young’s inequality gives

$$|\langle f, y_h \rangle| \leq \nu |y_h|_1^2 + \frac{1}{4\nu} \| f \|_{s,h}^2.$$ 

Thus, another application of (4) eliminates the term $\nu |y_h|_1^2$ and we are left with

$$\| \Pi_h^* (\nabla p_h) \|_{\tau_p}^2 + \| \Pi_h^* (u \cdot \nabla y_h) \|_{\tau_v}^2 \leq \frac{1}{4 \nu} \| f \|_{s,h}^2, \quad (44)$$

which is slightly sharper than (41).

(2) To obtain the pressure estimate, we use the inf–sup condition (31):

$$C \| p_h \|_0 \leq \sup_{v_h \in Y_h} \frac{\langle \nabla \cdot v_h, p_h \rangle}{|v_h|_1} + \| \Pi_h^* (\nabla p_h) \|_{\tau_p} + \left( \sum_{i=1}^{R} \left( \sup_{w_h \in Y_h(\mathcal{O}_i)} \frac{\langle \nabla \cdot w_h, p_h \rangle_{\mathcal{O}_i}}{|w_h|_1,\mathcal{O}_i} \right)^2 \right)^{1/2} \quad (46)$$

$$= I + II + III.$$ 

The second term is bounded by (44)

$$|II| \leq \frac{1}{2 \sqrt{\nu}} \| f \|_{s,h}.$$ 

To bound the first term, let us take $q_h = 0$ in (7):

$$\langle \nabla \cdot v_h, p_h \rangle = a(y_h, v_h) + \sum_{K \in T_h} \tau_K (\Pi_h^* (u \cdot \nabla y_h), \Pi_h^* (u \cdot \nabla v_h))_K - \langle f, v_h \rangle. \quad (45)$$
Owing to (44), the stabilizing term has the bound

\[
\sum_{K \in T_h} \tau_{vK} (\Pi_h^*(u \cdot \nabla v_h), \Pi_h^*(u \cdot \nabla v_h))_K \leq \frac{1}{2\sqrt{\nu}} \| f \|_{s,h} \| \Pi_h^*(u \cdot \nabla v_h) \|_{\tau_v}. \tag{46}
\]

Now, in view of (22),

\[
\| \Pi_h^*(u \cdot \nabla v_h) \|_{\tau_v}^2 \leq C_2 \sum_{i=1}^R h_i^2 \| \Pi_h^*(u \cdot \nabla v_h) \|_{0,\omega_i}^2,
\]

with \( r^* > 2 \) defined by \( 1/r + 1/r^* = 1/2 \). But (11) applied to \( \nabla v_h \) in each \( T \) of \( \omega_K \) gives

\[
\| \nabla v_h \|_{0,r^*,\omega_K}^2 \leq \hat{C} \sum_{T \subset \omega_K} h_T^{2d/r^*} \| \nabla v_h \|_{0,T}^2.
\]

Therefore, applying (21), we obtain

\[
\| \Pi_h^*(u \cdot \nabla v_h) \|_{\tau_v}^2 \leq C \sum_{i=1}^R h_i^2 \| u \|_{0,r,\omega_i}^2 \| \nabla v_h \|_{0,\omega_i}^2,
\]

and again (21) implies

\[
h_i^2 \frac{h_i^{2d/r^*}}{\rho_i^d} = h_i^{2-d+2d/r^*} \left( \frac{h_i}{\rho_i} \right)^d \leq Ch_i^{2(1-d/r)},
\]

with \( 1 - d/r > 0 \) since \( r > d \). Hence

\[
\| \Pi_h^*(u \cdot \nabla v_h) \|_{\tau_v}^2 \leq Ch^{2(1-d/r)} \sum_{i=1}^R \sum_{K \subset \Omega_i} \| u \|_{0,r,\omega_i}^2 \| \nabla v_h \|_{0,\omega_i}^2 \leq Ch^{2(1-d/r)} \| u \|_{0,r}^2 \| \nabla v_h \|_{0}^2.
\]

Then the fact that an element belongs to at most \( M \) macro-elements and the definition of \( \omega_K \) imply

\[
\| \Pi_h^*(u \cdot \nabla v_h) \|_{\tau_v} \leq Ch^{1-d/r} \| u \|_{0,r} \| \nabla v_h \|_{0}, \tag{48}
\]

and substituting (48) into (46) gives

\[
\sum_{K \in T_h} \tau_{vK} (\Pi_h^*(u \cdot \nabla v_h), \Pi_h^*(u \cdot \nabla v_h))_K \leq \frac{C}{\sqrt{\nu}} \| f \|_{s,h} h^{1-d/r} \| u \|_{0,r} \| \nabla v_h \|_{0}. \tag{49}
\]
To estimate the remaining terms in (45), we use the continuity of form $a$ and (43):

$$|a(y_h,v_h) - (f,v_h)| \leq C((v + \|u\|_{0,r})|y_h|_1 + \|f\|_{s,h})|v_h|_1 \leq C\left(1 + \frac{\|u\|_{0,r}}{v}\right)\|f\|_{s,h}|v_h|_1. \quad (50)$$

Finally, by substituting (49) and (50) into (45), we obtain

$$|I| \leq C\left(1 + \frac{\|u\|_{0,r}}{\sqrt{v}} + \frac{\|u\|_{0,r}}{v}\right)\|f\|_{s,h}. \quad (51)$$

It remains to estimate $\text{III}$. With obvious notation, problem (7) with $q_h = 0$ and $v_h = w_h$, for $w_h \in Y_h(\mathcal{O}_i)$, yields

$$(\nabla \cdot w_h,p_h)_{\mathcal{O}_i} = a(y_h,w_h)_{\mathcal{O}_i} + \sum_{K \subset \mathcal{O}_i} \tau_{iK}(\Pi_h^*(u \cdot \nabla y_h),\Pi_h^*(u \cdot \nabla w_h))_K - (f,w_h)_{\mathcal{O}_i}. \quad (52)$$

Note that the scalar products are restricted to the support of $w_h$, i.e., $\mathcal{O}_i$. For the stabilization term, we write, again with obvious notation

$$\left|\sum_{K \subset \mathcal{O}_i} \tau_{iK}(\Pi_h^*(u \cdot \nabla y_h),\Pi_h^*(u \cdot \nabla w_h))_K\right| \leq \|\Pi_h^*(u \cdot \nabla y_h)\|_{\tau_i,\mathcal{O}_i} \|\Pi_h^*(u \cdot \nabla w_h)\|_{\tau_i,\mathcal{O}_i}.$$

The first factor is estimated as above:

$$\|\Pi_h^*(u \cdot \nabla y_h)\|_{\tau_i,\mathcal{O}_i} \leq Ch_i^{1-d/r}\left(\sum_{K \subset \mathcal{O}_i} \|u\|^2_{0,r,\omega_K} \|\nabla y_h\|^2_{0,\omega_K}\right)^{1/2}. \quad (53)$$

Let

$$D_i = \bigcup_{K \subset \mathcal{O}_i} \omega_K.$$ 

Then

$$\|\Pi_h^*(u \cdot \nabla y_h)\|_{\tau_i,\mathcal{O}_i} \leq Ch_i^{1-d/r}\|u\|_{0,r,D_i} \|\nabla y_h\|_{0,D_i}. \quad (54)$$

The second factor is simpler because $w_h$ vanishes outside $\mathcal{O}_i$ and therefore $\Pi_h$ only uses values of $u \cdot \nabla w_h$ in $\overline{\mathcal{O}_i}$. Thus, we have the local version of (48):

$$\|\Pi_h^*(u \cdot \nabla w_h)\|_{\tau_i,\mathcal{O}_i} \leq Ch_i^{1-d/r}\|u\|_{0,r,\mathcal{O}_i} \|\nabla w_h\|_{0,\mathcal{O}_i}. \quad (55)$$

Hence (53) and (54) imply

$$\left|\sum_{K \subset \mathcal{O}_i} \tau_{iK}(\Pi_h^*(u \cdot \nabla y_h),\Pi_h^*(u \cdot \nabla w_h))_K\right| \leq Ch_i^{2-2d/r}\|u\|_{0,r,\mathcal{O}_i} \|u\|_{0,r,\mathcal{O}_i} \|\nabla y_h\|_{0,D_i} \|\nabla w_h\|_{0,\mathcal{O}_i}.$$ 

Collecting these bounds and using the local continuity of form $a$, we obtain

$$|\nabla \cdot w_h,p_h\rangle_{\mathcal{O}_i} \leq C(v + \|u\|_{0,r,\mathcal{O}_i} + h_i^{2-2d/r}\|u\|_{0,r,\mathcal{O}_i} \|u\|_{0,r,\mathcal{O}_i})|y_h|_{1,\mathcal{O}_i}|w_h|_{1,\mathcal{O}_i}$$

$$+ \|f\|_{-1,\mathcal{O}_i}|w_h|_{1,\mathcal{O}_i}.$$
Squaring and dividing by $|w_h|^2_{1,\Omega}$, and using the fact that both $\|u\|_{0,r,\mathcal{O}_i}$ and $\|u\|_{0,r,D_i}$ are bounded by $\|u\|_{0,r}$, we deduce that

$$\sup_{w \in \mathcal{Y}_h(\mathcal{O}_i)} \frac{|(\nabla \cdot w_h, u_h)_{\mathcal{O}_i}|^2}{|w_h|^2_{1,\Omega}^2} \leq C(v^2 + \|u\|^2_{0,r,\mathcal{O}_i} + h^{4(1-d/r)}\|u\|^2_{0,r,\mathcal{O}_i} \|u\|^2_{0,r,D_i}) |y_h|^2_{1,D_i} + C\|f\|^2_{-1,\Omega}$$

which implies

$$\leq C(v^2 + \|u\|^2_{0,r} + h^{4(1-d/r)}\|u\|^4_{0,r}) |y_h|^2_{1,\Omega} + C\|f\|^2_{-1,\Omega}.$$  

After summing over all macro-elements and using the fact that an element belongs to at most $M$ macro-elements and the definition of $D_i$, this inequality becomes

$$\sum_{i=1}^R \sup_{w \in \mathcal{Y}_h(\mathcal{O}_i)} \frac{|(\nabla \cdot w_h, u_h)_{\mathcal{O}_i}|^2}{|w_h|^2_{1,\Omega}^2} \leq C(v^2 + \|u\|^2_{0,r} + h^{4(1-d/r)}\|u\|^4_{0,r}) |y_h|^2_{1,\Omega} + C\sum_{i=1}^R \|f\|^2_{-1,\Omega}.$$  

Therefore

$$|III| \leq C(v + \|u\|_{0,r} + h^{2(1-d/r)}\|u\|^2_{0,r}) |y_h|^2_{1} + C\left( \sum_{i=1}^R \|f\|^2_{-1,\Omega} \right)^{1/2}.$$  

Then, by substituting the bound (43) for $y_h$ into this inequality, we obtain

$$|III| \leq C(v + \|u\|_{0,r} + h^{2(1-d/r)}\|u\|^2_{0,r}) \frac{1}{\nu} |f|_{\nu,h} + C\left( \sum_{i=1}^R \|f\|^2_{-1,\Omega} \right)^{1/2}. \quad (55)$$

Finally (42) follows by substituting (55), the bound for $II$, and (51) into (31).

### 3.3 Error estimates

**Theorem 3.9** In addition to the assumptions of Theorem 3.8, we suppose that Hypothesis 2.4 holds, that $u \in H^s(\Omega)^d$ with $s > \ell - 1$ and that the solution of problem (5) verifies $(y, p) \in H^{\ell+1}(\Omega)^d \times H^m(\Omega)$. Then the following error estimates hold for some constant $C > 0$ independent of $h$ and $v$:

$$|y - y_h|_1 \leq C(A_1 h^\ell + B_1 h^m + D_1 h^k), \quad (56)$$

$$\|p - p_h\|_0 \leq C((v + \|u\|_{0,r}) |y - y_h|_1 + A_2 h^\ell + B_2 h^m + D_2 h^k), \quad (57)$$

where $k = \min\{m, \ell\}$, and

- $A_1 = |y|_{\ell+1}(1 + (1 + \|u\|_s)/\sqrt{v} + (1/v + 1/\sqrt{v})h^{1-d/r}\|u\|_{0,r,g})$.
- $B_1 = |p|_m(1/v + 1/\sqrt{v})$.
- $D_1 = |p|_{\ell}(1/\sqrt{v})$.
- $A_2 = |y|_{\ell+1}(1 + \|u\|_s + \sqrt{v} + h^{1-d/r}\|u\|_{0,r,\mathcal{O}}(1 + \|u\|_s + (1 + 1/\sqrt{v})(1 + h^{1-d/r}\|u\|_{0,r}))$.
- $B_2 = |p|_m((1 + h^{1-d/r}\|u\|_{0,r})(1 + 1/\sqrt{v}))$.
- $D_2 = |p|_{\ell}(1 + h^{1-d/r}\|u\|_{0,r})$.

**Proof.** All constants $C$ below are independent of $h$ and $v$. 


In view of Sobolev’s imbeddings, we observe that, whatever the value of \( \ell \geq 2 \), the regularity assumption on \( u \) implies that \( u \in L^\infty(\Omega)^d \) when \( d = 2 \) and \( u \in L^t(\Omega)^d \) for some \( t > 6 \) when \( d = 3 \). Hence \( r = \infty \) when \( d = 2 \) and \( r > 6 \) when \( d = 3 \).

For \( 1 \leq t \leq m \), the standard finite element interpolation operator \( T_h : H^t(\Omega) \cap L^2_0(\Omega) \to M_h \) satisfies for \( k = 0, 1 \), with some constant \( C > 0 \) independent of \( h \),

\[
\forall K \in T_h, \forall q \in H^t(\Omega) \cap L^2_0(\Omega), \quad \| q - T_h q \|_{k,K} \leq C h^{\ell - k} |q|_{\ell,\omega_K}, \tag{58}
\]

where \( \omega_K \) can be replaced by \( K \) when \( t > d/2 \). Of course, the local quasi-uniformity of the mesh implies also the same global accuracy.

Let \( (y, p) \) be the unique solution of problem (5) and \( (y_h, p_h) \) be the unique solution of (7), and set \( U_h = R_h y, P_h = T_h p \). We define \( s_h \in H^{-1}(\Omega)^d \) by

\[
\langle s_h, v \rangle = a(U_h, v) - (P_h, \nabla \cdot v) - \langle f, v \rangle, \quad \forall v \in H^1_0(\Omega)^d. \tag{59}
\]

By subtracting (5) with \( q = 0 \) from this equality, we obtain

\[
\langle s_h, v \rangle = a(U_h - y, v) - (P_h - p, \nabla \cdot v), \quad \forall v \in H^1_0(\Omega)^d. \tag{60}
\]

To estimate \( \| s_h \|_{-1} \) from (60), we apply Green’s formula to the convection term in \( a \):

\[
| (u \cdot \nabla (U_h - y), v) | = | - (u \cdot \nabla v, U_h - y) | \leq \| u \|_{0,r} \| U_h - y \|_{0,r} \| v \|_1, \tag{61}
\]

with \( 1/r + 1/r^* = 1/2 \), \( r^* = 2 \) when \( d = 2 \) and \( r^* < 3 \) when \( d = 3 \). Hence the interpolation error estimates (37) and (58) lead to

\[
\| s_h \|_{-1} = \sup_{v \in H^1_0(\Omega)^d} \frac{|\langle s_h, v \rangle|}{|v|_1} \leq \| U_h - y \|_1 + \| u \|_{0,r} \| U_h - y \|_{0,r} + \| P_h - p \|_0 \\
\leq C (\| U_h \|_1 + h^{\ell + 1 - d/r} \| y \|_{\ell+1} \| u \|_{0,r} + H^m \| p \|_m). \tag{62}
\]

Consider the errors \( e_h = y_h - U_h, \lambda_h = p_h - P_h \). By using (7) and (59), we obtain the following error equation: For all \( (v_h, q_h) \in Y_h \times M_h \),

\[
L(e_h, \lambda_h; v_h, q_h) + (\nabla \cdot U_h, q_h) + \langle s_h, v_h \rangle + \sum_{K \in T_h} \tau_{p_K} (\Pi_h^*(\nabla p_h), \Pi_h^*(\nabla q_h))_K \\
+ \sum_{K \in T_h} \tau_{v_K} (\Pi_h^*(u \cdot \nabla y_h), \Pi_h^*(u \cdot \nabla v_h))_K = 0. \tag{63}
\]

This equation has the form

\[
L_h(e_h, \lambda_h; v_h, q_h) = \langle G_h, (v_h, q_h) \rangle,
\]

with

\[
\langle G_h, (v_h, q_h) \rangle = - (\nabla \cdot U_h, q_h) - \langle s_h, v_h \rangle - \sum_{K \in T_h} \tau_{p_K} (\Pi_h^*(\nabla p_h), \Pi_h^*(\nabla q_h))_K \\
- \sum_{K \in T_h} \tau_{v_K} (\Pi_h^*(u \cdot \nabla U_h), \Pi_h^*(u \cdot \nabla v_h))_K.
\]
Since $G_h$ acts on both $v_h$ and $q_h$, we cannot derive the error estimates by applying directly Theorem 3.8 to (63), but the proof follows the same pattern.

(1) The choice $v_h = e_h$ and $q_h = \lambda h$ in (63) gives

$$\alpha(e_h, e_h) + \| \Pi_h^n(\nabla \lambda h) \|^2_{\tau_p} + \| \Pi_h^n(u \cdot \nabla e_h) \|^2_{\tau_p} = - (\nabla \cdot (U_h - y), \lambda h) - (s_h, e_h)$$

$$- (\Pi_h^n(\nabla P_h), \Pi_h^n(\nabla \lambda h))_{\tau_p} - (\Pi_h^n(u \cdot \nabla U_h), \Pi_h^n(u \cdot \nabla e_h))_{\tau_v},$$

where we have used the fact that $\nabla \cdot y = 0$. But by Green’s formula and (36)

$$- (\nabla \cdot (U_h - y), \lambda h) = (U_h - y, \nabla \lambda h) = (U_h - y, \Pi_h^n(\nabla \lambda h)).$$

Therefore

$$|(\nabla \cdot (U_h - y), \lambda h)| \leq \sum_{k \in T_h} \| U_h - y \|_{0,K} \| \Pi_h^n(\nabla \lambda h) \|_{0,K}$$

$$\leq \| \Pi_h^n(\nabla \lambda h) \|_{\tau_p} \left( \sum_{k \in T_h} \frac{1}{\tau_{pK}} \| U_h - y \|^2_{0,K} \right)^{1/2}$$

$$\leq \frac{1}{C} \| \Pi_h^n(\nabla \lambda h) \|_{\tau_p} \left( \sum_{i=1}^R \frac{1}{h_i^2} \| U_h - y \|^2_{0,\mathcal{O}} \right)^{1/2},$$

with the constant $C^2 = \alpha_1 C^2_1$ of (8) and (21). Then the ellipticity (4) of $\alpha$, this bound, (62), and several applications of Young’s inequality easily lead to a first estimate:

$$v_1 |e_h|^2 + \| \Pi_h^n(\nabla \lambda h) \|^2_{\tau_p} + \| \Pi_h^n(u \cdot \nabla e_h) \|^2_{\tau_p} \leq \sum_{i=1}^R \frac{1}{h_i^2 C^2} \| U_h - y \|^2_{0,\mathcal{O}} + 2 \| \Pi_h^n(\nabla P_h) \|^2_{\tau_p}$$

$$+ \| \Pi_h^n(u \cdot \nabla U_h) \|^2_{\tau_v} + 3v \| U_h - y \|^2_1 + \frac{3}{v} \| u \|^2_{0,\mathcal{O}} \| U_h - y \|^2_{0,\mathcal{O}} + \frac{3}{v} \| P_h - v \|^2_0. \hspace{1cm} (64)$$

(2) To estimate $\lambda h$, we take $q_h = 0$ and $v_h$ arbitrary in (63):

$$(\lambda h, \nabla \cdot v_h) = \alpha(e_h, v_h) + (\Pi_h^n(u \cdot \nabla e_h), \Pi_h^n(u \cdot \nabla v_h))_{\tau_v} + (s_h, v_h)$$

$$+ (\Pi_h^n(u \cdot \nabla U_h), \Pi_h^n(u \cdot \nabla v_h))_{\tau_v}. \hspace{1cm} (65)$$

The continuity of $\alpha$ on one the hand and (48) on the other give

$$|\alpha(e_h, v_h)| \leq (v + C \| u \|_{0,\mathcal{O}}) |e_h|_1 |v_h|_1,$$

$$\| \Pi_h^n(u \cdot \nabla v_h) \|_{\tau_v} \leq C h^{1-d/r} \| u \|_{0,\mathcal{O}} |v_h|_1.$$. 
Then considering (62), the first term of the inf–sup condition (31) satisfied by \( \lambda_h \) is bounded by

\[
\sup_{v_h \in Y_h} \frac{(\nabla \cdot v_h, \lambda_h)}{|v_h|_1} \leq (v + C\|u\|_{0,r})|e_h|_1 + Ch^{1-d/r}\|u\|_{0,r}(\|\Pi_h^*(u \cdot \nabla e_h)\|_{r_1} + \|\Pi_h^*(u \cdot \nabla U_h)\|_{r_1}) + v|U_h - y|_1 + \|u\|_{0,r}\|U_h - y\|_{0,r} + \|P_h - p\|_0.
\]

(66)

Arguing as in the proof of Theorem 3.8, we see that the last term of (31) satisfies a similar bound. Indeed, we have with obvious notation

\[
\forall w_h \in Y_h(\mathcal{O}_i), \quad (\nabla \cdot w_h, \lambda_h)_{\mathcal{O}_i} = a(e_h, w_h)_{\mathcal{O}_i} + (\Pi_h^*(u \cdot \nabla e_h), \Pi_h^*(u \cdot \nabla w_h))_{\mathcal{O}_i} + (\Pi_h^*(u \cdot \nabla U_h), \Pi_h^*(u \cdot \nabla w_h))_{\mathcal{O}_i}.
\]

(67)

As in (54), we have

\[
\|\Pi_h^*(u \cdot \nabla w_h)\|_{r_1,\mathcal{O}_i} \leq Ch^{1-d/r}\|u\|_{0,r,\mathcal{O}_i}\|\nabla w_h\|_{0,\mathcal{O}_i}.
\]

Then we infer from (67)

\[
\sup_{w_h \in Y_h(\mathcal{O}_i)} \frac{(\nabla \cdot w_h, \lambda_h)}{|w_h|_{1,\mathcal{O}_i}} \leq (v + C\|u\|_{0,r,\mathcal{O}_i})|e_h|_{1,\mathcal{O}_i} + Ch^{1-d/r}\|u\|_{0,r,\mathcal{O}_i}(\|\Pi_h^*(u \cdot \nabla e_h)\|_{r_1,\mathcal{O}_i} + \|\Pi_h^*(u \cdot \nabla U_h)\|_{r_1,\mathcal{O}_i}) + v|U_h - y|_{1,\mathcal{O}_i} + \|u\|_{0,r,\mathcal{O}_i}\|U_h - y\|_{0,r,\mathcal{O}_i} + \|P_h - p\|_{0,\mathcal{O}_i}.
\]

Therefore squaring, summing over \( i \), using the fact that an element can belong to at most \( M \) macro-elements \( \mathcal{O}_i \) and taking the square root, we obtain

\[
\left( \sum_{i=1}^R \sup_{w_h \in Y_h(\mathcal{O}_i)} \frac{(\nabla \cdot w_h, \lambda_h)_{\mathcal{O}_i}}{|w_h|_{1,\mathcal{O}_i}} \right)^{1/2} \leq C \left( (v + \|u\|_{0,r})|e_h|_1 + \|P_h - p\|_0 + v|U_h - y|_1 + \|u\|_{0,r} \left( \sum_{i=1}^R \|U_h - y\|_{0,r,\mathcal{O}_i}^2 \right)^{1/2} + h^{1-d/r}(\|\Pi_h^*(u \cdot \nabla e_h)\|_{r_1} + \|\Pi_h^*(u \cdot \nabla U_h)\|_{r_1}^2) \right).
\]

(68)

By substituting (66) and (68) into (31), we derive an intermediate bound for \( \lambda_h \):

\[
\|\lambda_h\|_0 \leq C \left( (v + \|u\|_{0,r})|e_h|_1 + \|P_h - p\|_0 + v|U_h - y|_1 + \|\Pi_h^*(\nabla \lambda_h)\|_{r_1} + \|\Pi_h^*(u \cdot \nabla U_h)\|_{r_1} \right) \left( \sum_{i=1}^R \|U_h - y\|_{0,r,\mathcal{O}_i}^2 \right)^{1/2} + h^{1-d/r}(\|\Pi_h^*(u \cdot \nabla e_h)\|_{r_1} + \|\Pi_h^*(u \cdot \nabla U_h)\|_{r_1}^2) \right).
\]

(69)

(3) Let us estimate the terms in the right-hand sides of (64) and (69). As far as the interpolation errors are concerned, owing to (38), (58), and again the fact that an element can belong to at most \( M \) macro-elements \( \mathcal{O}_i \) and taking the square root, we obtain

\[
\left( \sum_{i=1}^R \sup_{w_h \in Y_h(\mathcal{O}_i)} \frac{(\nabla \cdot w_h, \lambda_h)_{\mathcal{O}_i}}{|w_h|_{1,\mathcal{O}_i}} \right)^{1/2} \leq C \left( (v + \|u\|_{0,r})|e_h|_1 + \|P_h - p\|_0 + v|U_h - y|_1 + \|u\|_{0,r} \left( \sum_{i=1}^R \|U_h - y\|_{0,r,\mathcal{O}_i}^2 \right)^{1/2} + h^{1-d/r}(\|\Pi_h^*(u \cdot \nabla e_h)\|_{r_1} + \|\Pi_h^*(u \cdot \nabla U_h)\|_{r_1}^2) \right).
\]

(68)
macro-elements, we have, respectively,

\[
|U_h - y|_1 + \left( \sum_{i=1}^{R} \frac{1}{h_i} \|U_h - y\|_{0,\Omega_i}^2 \right)^{1/2} \leq Ch^{\ell}|y|_{\ell+1}, \quad \|P_h - p\|_0 \leq Ch^m|p|_m,
\]

\[
\left( \sum_{i=1}^{R} \|U_h - y\|_{0,\Omega_i}^2 \right)^{1/2} \leq Ch^{\ell+1-d/2} \left( \sum_{i=1}^{R} |y|_{\ell+1,\Omega_i}^2 \right)^{1/2} \leq Ch^{\ell+1-d/2}|y|_{\ell+1}.
\]

To estimate the term \( \|\Pi_h^s(\nabla P_h)\|_{r_p} \), we add and subtract \( \nabla p \); this yields

\[
\|\Pi_h^s(\nabla P_h)\|_{r_p} \leq \|\Pi_h^s(\nabla (P_h - p))\|_{r_p} + \|\Pi_h^s(\nabla p)\|_{r_p} \\
\leq Ch\|\nabla (P_h - p)\|_0 + Ch\|\Pi_h^s(\nabla p)\|_0 \\
\leq C(h^m|p|_m + h^k|p|_k),
\]

where \( k = \min\{m, \ell\} \). The last inequality is a consequence of error estimates (58) and (15).

To estimate \( \|\Pi_h^s(u \cdot \nabla U_h)\|_{r_s} \), we add and subtract \( u \cdot \nabla y \):

\[
\|\Pi_h^s(u \cdot \nabla U_h)\|_{r_s} \leq \|\Pi_h^s(u \cdot \nabla (U_h - y))\|_{r_s} + \|\Pi_h^s(u \cdot \nabla y)\|_{r_s}.
\]

For the first term, we use (22) and Hypothesis 2.3:

\[
\|\Pi_h^s(u \cdot \nabla (U_h - y))\|_{r_s} \leq Ch\|\Pi_h^s(u \cdot \nabla (U_h - y))\|_0 \leq Ch\|u\|_{0,\Omega} \|\nabla (U_h - y)\|_{0,\Omega}.
\]

Applying now (38), we obtain

\[
\|\Pi_h^s(u \cdot \nabla (U_h - y))\|_{r_s} \leq Ch^{\ell+1-d/2}\|u\|_{0,\Omega} |y|_{\ell+1}.
\]

For the second term in (71), we use Theorem 1.4.4.2 of Grisvard (see Grisvard, 1985); concretely if \( u \in H^s(\Omega)^d \) with \( s > \max\{\ell - 1, d/2 - 1\} \) and \( y \in H^{\ell+1}(\Omega)^d \), then \( u \cdot \nabla y \in H^{\ell-1}(\Omega)^d \) and

\[
\|u \cdot \nabla y\|_{\ell-1} \leq C\|u\|_s \|y\|_{\ell+1}.
\]

Here \( \ell \geq 2 \), thus it suffices that \( s > \ell - 1 \). Then, using (22) and (15),

\[
\|\Pi_h^s(u \cdot \nabla y)\|_{r_s} \leq Ch\|\Pi_h^s(u \cdot \nabla y)\|_0 \leq Ch^{\ell}\|u \cdot \nabla y\|_{\ell-1} \leq Ch^{\ell}\|u\|_s \|y\|_{\ell+1}.
\]

(4) Finally (56) and (57) are derived by substituting these inequalities into (64) and (69).

\[\square\]

**Remark 3.10**

(1) The above estimates are optimal for moderate convection or diffusion-dominated flows. In this case the optimal choice for the interpolation of velocity and pressure is \( m = \ell \). However, for convection-dominated flows, these estimates deteriorate unless \( h \) is very small.

(2) Note that if \( y \) belongs to \( H^{\ell+1}(\Omega)^d \), then in general \( p \) can be at most in \( H^{\ell}(\Omega) \), therefore the assumption \( p \in H^{\ell+1}(\Omega) \) is not realistic. Thus the choice \( m = \ell \) is consistent with the regularity of \( y \).
4. An example of nodal interpolant

In this section, we briefly analyse a variant of (7) that uses for \( \Pi_h \) an extension of the nodal interpolant introduced by Girault, Kanschat and Rivière in Girault et al. (2012). We shall see that the previous estimates carry over to this interpolant, up to some modifications described below.

For any integer \( k \geq 0 \), let

\[
X_h = \{ v_h \in L^2(\Omega); \forall K \in \mathcal{T}_h, v_h|_K \in \mathbb{P}_k(K) \}.
\]

We want to define \( \Pi_h : X_h \mapsto V_{\ell-1}^h \) such that \( \Pi_h \) coincides with the standard nodal Lagrange interpolant when acting on continuous functions. For any \( K \in \mathcal{T}_h \), let \( \Sigma_K \) denote its principal lattice of order \( \ell - 1 \) and let

\[
\mathcal{N} = \bigcup_{K \in \mathcal{T}_h} \Sigma_K.
\]

For any \( K \) and for any \( a \in \Sigma_K \), let \( \varphi_a \in \mathbb{P}_{\ell-1}(K) \) be the standard basis function associated with \( a \), i.e.,

\[
\forall a, b \in \Sigma_K, \quad \varphi_a(b) = \delta_{a,b},
\]

\( \varphi_a \) is supported by the union of elements sharing \( a \) and extended by 0 outside this union. For any \( a \in \mathcal{N} \), if \( a \in \bar{K} \) for some \( K \), we set

\[
\Pi_h(v_h)(a) = v_h(a). \tag{74}
\]

Otherwise, if \( a \in \partial K \) for some \( K \), we associate with \( a \) an element \( K_a \) whose boundary contains \( a \), and we set

\[
\Pi_h(v_h)(a) = v_h|_{K_a}(a). \tag{75}
\]

From now on, we assume that the choice \( K_a \) is fixed. Then we define

\[
\Pi_h(v_h)(x) = \sum_{a \in \mathcal{N}} \Pi_h(v_h)(a) \varphi_a(x). \tag{76}
\]

By construction \( \Pi_h(v_h) \) belongs to \( V_{\ell-1}^h \) and \( \Pi_h(v) \) reduces to the standard nodal Lagrange interpolant \( I_{\ell-1}^h (v) \) when \( v \) belongs to \( C^0(\Omega) \). This interpolant is similar to the one by Scott-Zhang in the sense that both are Lagrange interpolants with nodal values obtained by averaging on a single element. Here averaging is reduced to just the nodal value.

Then, we define \( L_h(y, p; v, q) \) in (7) by

\[
L_h(y, p; v, q) = L(y, p; v, q) + \sum_{K \in \mathcal{T}_h} \tau_{pK}(\Pi_h^*(\nabla p), \Pi_h^*(\nabla q))_K
\]

\[
+ \sum_{K \in \mathcal{T}_h} \tau_{qK}(\Pi_h^*(u_h \cdot \nabla y), \Pi_h^*(u_h \cdot \nabla v))_K. \tag{77}
\]

Here \( u_h \) is globally continuous; it can be a discrete approximation of \( u \) possibly obtained from a previous computation, which is frequently the case in practice; or it can be some interpolant of \( u \).

In order to establish the stability and approximation properties of \( \Pi_h \), we define the set \( \omega_e \) of interfaces between elements contained in \( \omega_K \), we introduce the jump of \( v \) through any interface \( e \) between
elements $K_1$ and $K_2$:

$$[v]_e = (v|_{K_1} - v|_{K_2})|_e,$$

and we denote by $h_e$ the diameter of $e$.

**Lemma 4.1** Under Hypothesis 2.2, there exists a constant $C$, independent of $h$ and $K$, such that

$$\forall v_h \in X_h, \forall K \in T_h, \quad \|\Pi_h(v_h) - v_h\|_{0,K} \leq Ch_K \left( \|\nabla v_h\|_{0,K}^2 + \sum_{e \in \partial K} \frac{1}{h_e} \|[v_h]_e\|_{0,e}^2 \right)^{1/2}.$$  (78)

**Proof.** The proof is similar to that of Girault et al. (2012, Lemma 15). By definition, we have in each $K$,

$$\|\Pi_h(v_h) - v_h\|^2_{L^2(K)} = \int_K \left| \sum_{a \in \Sigma} \Pi_h(v_h)(a)\varphi_a(x) - v_h(x) \right|^2$$

$$= \int_K \left| \sum_{a \in \Sigma} (\Pi_h(v_h)(a) - v_h(x))\varphi_a(x) \right|^2,$$

because

$$\sum_{a \in \Sigma} \varphi_a(x)|_K = 1.$$ 

Then (74) and (75) imply

$$\|\Pi_h(v_h) - v_h\|^2_{0,K} = \int_K \left| \sum_{a \in \Sigma \cap K} (v_h(a) - v_h(x))\varphi_a(x) + \sum_{a \in \Sigma \cap \partial K} (v_h|_{K_a}(a) - v_h|_K(x))\varphi_a(x) \right|^2.$$ 

In the first sum, since both $a$ and $x$ belong to $K$ and $v_h$ is smooth in $K$, each term can be expressed by $\nabla v_h$. In the second sum, $K_a$ and $K$ are separated by a sequence of two-by-two adjacent elements of $\omega_K$ and at each common interface the difference in the values of $v_h$ is expressed by a jump. Then a straightforward computation yields (78). □

From (78), by passing to the reference element and applying inverse inequalities that are valid because $v_h$ in each $K$ belongs to a finite element space, we readily derive the analogue of (14):

$$\forall v_h \in X_h, \forall K \in T_h, \quad \|\Pi_h(v_h)\|_{0,K} \leq C\|v_h\|_{0,\omega_K}.$$  (79)

With this property, it is easy to prove (23) for all $g$ in $X_h$ and (24). The inf–sup condition (31) holds without modification. The statement of the stability Theorem 3.8 becomes the following theorem.
Theorem 4.2 Assume that (8) and Hypotheses 2.2 and 2.5 hold; then, problem (7), (77) has a unique solution. Moreover, there exists a constant $C > 0$, independent of $h$ and $v$, such that

$$v|y_h|_1 + \sqrt{v}\|\Pi_h^e(\nabla p_h)\|_{\tau_p} + \sqrt{v}\|\Pi_h^e(u_h \cdot \nabla y_h)\|_{\tau_p} \leq 2\|f\|_{\tau_p}.$$  

(80)

$$\|p_h\|_0 \leq C \left(1 + \frac{1}{\sqrt{v}} + \frac{|u|_{0,r}}{v} + \frac{|u_h|_{0,r}^2}{v}\right) \|f\|_{\tau_p} + C \left(\sum_{i=1}^R \|f\|_{\tau_p}^2\right)^{1/2}.$$  

(81)

Owing to (79), the proof follows immediately from that of Theorem 3.8 by replacing $u$ by $u_h$ in the stabilizing term.

The proof of the error estimates is similar, except for the bounds of the two terms: $\|\Pi_h^e(\nabla p_h)\|_{\tau_p}$ and $\|\Pi_h^e(u_h \cdot \nabla U_h)\|_{\tau_p}$, where $P_h = T_h p$ and $U_h = R_h y$, $T_h$ satisfying (58) and $R_h$ satisfying (38). Consider the first term; the argument depends on the value of $m$.

Proposition 4.3 Let $m \geq 3$ and $k = \min(m, \ell)$. Under Hypotheses 2.2 and 2.4, and if (8) holds, there exists a constant $C$, independent of $h$, such that

$$\forall p \in H^m(\Omega), \quad \|\Pi_h^e(\nabla T_h(p))\|_{\tau_p} \leq C(h^m|p|_m + h^\ell|p|_k).$$  

(82)

Proof. If $m \geq 3$, then $\nabla p$ is continuous and $\Pi_h(\nabla p) = I_h^{\ell-1}(\nabla p)$. Therefore

$$\Pi_h^e(\nabla p_h) = \Pi_h^e(\nabla T_h p - I_h^{\ell-1}(\nabla p)).$$

Hence we can apply the stability bound (79) to $v_h = \nabla T_h p - I_h^{\ell-1}(\nabla p)$. This gives

$$\|\Pi_h^e(\nabla p_h)\|_{0,K} \leq C\|\nabla T_h p - I_h^{\ell-1}(\nabla p)\|_{0,K} \leq C(\|\nabla (T_h p - p)\|_{0,K} + \|\nabla p - I_h^{\ell-1}(\nabla p)\|_{0,K}).$$

The result follows by applying (58) and the approximation properties of $I_h^{\ell-1}$ in each $K$, together with the conditions stated in (8).

The above argument is not valid when $m = 2$ because $\nabla p$ is not continuous. Instead we apply (78) directly to $\nabla T_h p$ and observe that $\nabla p$ does not jump at interfaces because it belongs to $H^1(\Omega)$. Thus (79) gives

$$\|\Pi_h(\nabla T_h p) - \nabla T_h p\|_{0,K} \leq C h_K \left(\|D^2(T_h p)\|_{0,K}^2 + \sum_{e \in a_h} \frac{1}{h_e} \|[(\nabla (T_h p - p))_e\|_{0,e}^2\right)^{1/2}.$$  

The second sum is readily estimated by passing to the reference element, applying a trace theorem and using Hypothesis 2.2:

$$\frac{1}{h_e} \|[(\nabla (T_h p - p))_e\|_{0,e}^2 \leq \hat{C}|p|_{H^2(K_1 \cup K_2)}^2,$$

where $K_1$ and $K_2$ are the two elements sharing $e$. For the first sum, we use the fact that in each $K$, $T_h$ is uniformly stable in $H^2(K)$. Hence, if $m = 2$, Hypothesis 2.2 and (8) imply that

$$\forall p \in H^2(\Omega), \quad \|\Pi_h^e(\nabla p_h)\|_{\tau_p} \leq C h^2|p|_2.$$  

(83)

The treatment of the second term is simpler because the regularity assumptions on $y$ imply that $\nabla y$ is continuous. Recall that $\ell \geq 2$. 


Proposition 4.4 Let \( s > \ell - 1 \). Under Hypothesis 2.2 and (8), there exists a constant \( C \), independent of \( h \), such that
\[
\forall y \in H^{\ell+1}(\Omega), \quad \| \Pi_h^*(u_h \cdot \nabla R_h(y)) \|_{L^p} \leq C h^{\ell} \left( h^{1-d/r} \| u_h \|_{0,r} + \left( \sum_{k \in T_h} \| u_h \|_{2,K}^2 \right)^{1/2} \right) \| y \|_{\ell+1}. \tag{84}
\]

Proof. Since \( \ell + 1 \geq 3 \), it follows that \( u_h \cdot \nabla y \) is continuous and \( \Pi_h(u_h \cdot \nabla y) = I_h^{\ell-1}(u_h \cdot \nabla y) \). Therefore
\[
\Pi_h^*(u_h \cdot \nabla R_h(y)) = \Pi_h^*(u_h \cdot \nabla R_h(y) - I_h^{\ell-1}(u_h \cdot \nabla y)).
\]

Hence we can apply the stability bound (79) to \( v_h = u_h \cdot \nabla R_h(y) - I_h^{\ell-1}(u_h \cdot \nabla y) \). This gives
\[
\| \Pi_h^*(u_h \cdot \nabla R_h(y)) \|_{0,K} \leq C \| u_h \cdot \nabla R_h(y) - I_h^{\ell-1}(u_h \cdot \nabla y) \|_{0,0K}
\]
\[
\leq C(\| u_h \cdot \nabla (R_h y - y) \|_{0,0K} + \| u_h \cdot \nabla y - I_h^{\ell-1}(u_h \cdot \nabla y) \|_{0,0K}).
\]

Then the estimates are done locally in each \( K \) as in the end of the proof of Theorem 3.9.

Thus we have the analogue of Theorem 3.9.

Theorem 4.5 In addition to the assumptions of Theorem 4.2, we suppose that Hypothesis 2.4 holds, that the solution of problem (5) verifies \((y, p) \in H^{\ell+1}(\Omega)^d \times H^m(\Omega)\) and that \( (\sum_{K \in T_h} \| u_h \|_{2,K}^2)^{1/2} \) is uniformly bounded with \( s > \ell - 1 \). Then
\[
|y - y_h|_1 + \| p - p_h \|_0 \leq C(h^\ell + h^m + h^k),
\]
where \( k = \min\{m, \ell\} \).

Note that the above assumption on \( u_h \) is satisfied when \( u \in H^s(\Omega)^d \) and \( u_h \) is an interpolant of \( u \).

5. The case of a quasi-uniform mesh

In this short section, in addition to Hypothesis 2.2, we make the following assumption on the triangulation.

Hypothesis 5.1 The family of triangulations \( \{T_h\}_{h>0} \) is uniformly regular (also called quasi-uniform): there exists a constant \( \beta > 0 \), independent of \( h \) such that
\[
\forall K \in T_h, \quad \beta h \leq h_K \leq \sigma h_K. \tag{85}
\]

In this case, there is no need to argue locally. As a consequence, on the one hand, we can replace the local assumptions 2.3 and 2.4 on the operator \( \Pi_h \) by their global version:

Hypothesis 5.2 There exists a constant \( C > 0 \), independent of \( h \), such that
\[
\forall g \in L^2(\Omega)^d, \quad \| \Pi_h(g) \|_0 \leq \| g \|_0.
\]

Hypothesis 5.3 The operator \( \Pi_h \) satisfies the following optimal approximation error estimates. There exists a constant \( C \), independent of \( h \), such that
\[
\forall v \in W^{\ell,p}(\Omega)^d, \ 1 \leq p \leq +\infty, \quad \| v - \Pi_h(v) \|_{r,p} \leq C h^{\ell-r} |v|_{\ell,p}, \quad r = 0, 1, r \leq \ell. \tag{86}
\]
Both are also satisfied by the global $L^2(\Omega)$ orthogonal projection. And, on the other hand, we no longer need the extra smoothness (16) on the data; hence it suffices that $f$ belong to $H^{-1}(\Omega)^d$.

For the inf–sup condition, we observe that Lemma 3.1 is still valid, since it only relies on Hypothesis 2.2. Then the statement of Lemma 3.5 simplifies and we have the following inf–sup condition.

**Lemma 5.4** Let Hypotheses 2.2 and 5.1 hold and assume that the stabilization coefficients satisfy (8); then the following inf–sup condition holds:

$$\forall q_h \in M_h, \quad \|q_h\|_0 \leq C \left( \sup_{v_h \in Y_h} \frac{(\nabla \cdot v_h, q_h)}{|v_h|_1} + \|\Pi_h^s(\nabla q_h)\|_{\tau_p} \right).$$  \hspace{1cm} (87)

**Proof.** We start from (32), apply (33) and we must estimate $\|\Pi_h(\nabla q_h)\|_{\tau_p}$.

First, we observe that Hypothesis 5.1 on the triangulation and condition (8) on the stabilization coefficients imply

$$\forall z \in L^2(\Omega), \quad C_3 h \|z\|_0 \leq \|z\|_{\tau} \leq C_4 h \|z\|_0,$$  \hspace{1cm} (88)

with positive constants $C_3$ and $C_4$, independent of $h$, and where $\tau$ denotes either $\tau_v$ or $\tau_p$. In particular,

$$\|\Pi_h(\nabla q_h)\|_{\tau_p} \leq C h \|\Pi_h(\nabla q_h)\|_0.$$

As $\Pi_h(\nabla q_h) \in (V_h^{\ell-1})^d$, the inf–sup condition (26) yields

$$\|\Pi_h(\nabla q_h)\|_0 \leq C \sup_{v_h \in Y_h} \frac{(\Pi_h(\nabla q_h), v_h)}{|v_h|_0},$$

and consequently

$$\|\Pi_h(\nabla q_h)\|_{\tau_p} \leq C h \sup_{v_h \in Y_h} \frac{(\Pi_h(\nabla q_h), v_h)}{|v_h|_0}. \hspace{1cm} (89)$$

Again, the definition of $\Pi_h$ implies

$$|(\Pi_h(\nabla q_h), v_h)| \leq |(\nabla q_h, v_h)| + \|\Pi_h^s(\nabla q_h)\|_0 |v_h|_0.$$

Then (88) gives

$$\frac{|h(\Pi_h(\nabla q_h), v_h)|}{|v_h|_0} \leq C \left( \frac{|h(\nabla q_h, v_h)|}{|v_h|_0} + \|\Pi_h^s(\nabla q_h)\|_{\tau_p} \right).$$

Owing to Hypothesis 5.1, (10) implies that there exists a constant $C$ independent of $h$ such that

$$\forall v_h \in Y_h, \quad |v_h|_1 \leq C h^{-1} |v_h|_0. \hspace{1cm} (90)$$

Therefore,

$$h \sup_{v_h \in Y_h} \frac{(\Pi_h(\nabla q_h), v_h)}{|v_h|_0} \leq C \left( \sup_{v_h \in Y_h} \frac{(\nabla q_h, v_h)}{|v_h|_1} + \|\Pi_h^s(\nabla q_h)\|_{\tau_p} \right).$$

Then substituting into (89), this yields

$$\|\Pi_h(\nabla q_h)\|_{\tau_p} \leq C \left( \sup_{v_h \in Y_h} \frac{(\nabla q_h, v_h)}{|v_h|_1} + \|\Pi_h^s(\nabla q_h)\|_{\tau_p} \right),$$

and (87) follows by applying Green’s formula to the first term and using (32) and (33).
Remark 5.5 Here also Lemma 5.4 is still valid for Quadrilateral or Hexahedral finite elements if the family of triangulations is regular as in Remark 3.3.

Owing to this lemma, we have the same stability and error estimates as in the previous sections. We skip their proofs because they are a simplified version of the proofs of Theorems 3.8 and 3.9.

Theorem 5.6 Let Hypothesis 2.2 hold and let the stabilization coefficients satisfy (8). Then, under Hypotheses 5.1 and 5.2, problem (7) has a unique solution and, moreover, there exists a constant \(C > 0\) independent of \(h\) and \(\nu\) such that

\[
v |y_h|_1 + \sqrt{v} \| \Pi_h^* (\nabla p_h) \|_{\tau_p} + \sqrt{v} \| \Pi_h^* (u \cdot \nabla y_h) \|_{\tau_v} \leq 2 \| f \|_{*,h},
\]

\[
\| p_h \|_0 \leq C \left(1 + \frac{1}{\sqrt{v}} \| u \|_{0,r} (1 + \| u \|_{0,r}) \right) \| f \|_{*,h}.
\]

Theorem 5.7 In addition to the assumptions of Theorem 5.6, assume that Hypothesis 5.3 holds, that \(u\) is in \(H^s(\Omega)\) for some \(s > \ell - 1\), and that the solution \((y, p)\) of (5) has regularity \(H^{\ell+1}(\Omega)^d \times H^m(\Omega)\). Then the solution \((y_h, p_h)\) of (7) satisfies the error estimates (56) and (57) for some constant \(C\) independent of \(h\) and \(v\).

6. Numerical tests

We have performed several numerical tests of method (7) in 2D for testing the theoretical convergence order predicted by our analysis. For this we have considered an Oseen flow with a known smooth solution on uniformly regular and nonuniformly regular grids.

In all tests, we have used for \(\Pi_h\) the nodal interpolation operator defined by (76). Our computations have been performed with the free software FreeFem++ (cf. FreeFem++). We also have used the stabilization coefficients for convection given by (19) with \(C_1 = 2\) and \(C_2 = \sqrt{2}\), and the stabilization coefficients for pressure given by (20) with \(C_3 = 1/49\).

We have considered the Oseen problem (1) in the domain \(\Omega = [0, \pi] \times [0, \pi]\), with \(v = 10^{-2}\), and the following convection velocity and pressure:

- Horizontal velocity \(y_1 = 2(\sin x_1)^2 \sin x_2 \cos x_2\).
- Vertical velocity \(y_2 = -2 \sin x_1 (\sin x_2)^2 \cos x_1\).
- Pressure \(p = \cos x_1 \cos x_2\).

The data \(f\) is calculated to match these velocities and pressure.

For the grids considered, with \(h\) ranging from 0.4 to 0.09, the flow is convection-dominated, with maximum Péclet numbers ranging from 61 to 20. We represent in Table 1 the maximum and minimum Péclet number for the unstructured nonuniform meshes used, computed by (18) with \(r = 1\).

6.1 Test 1: Convergence order for uniformly regular meshes

We have made the calculations for both structured and unstructured meshes, with quite similar results in both cases. We present here the results for unstructured meshes.
Table 1 Minimum and maximum local Péclet numbers for nonuniformly regular meshes

<table>
<thead>
<tr>
<th>$h$</th>
<th>Min Péclet</th>
<th>Max Péclet</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.40</td>
<td>0.17</td>
<td>61.31</td>
</tr>
<tr>
<td>0.26</td>
<td>8.e−3</td>
<td>43.51</td>
</tr>
<tr>
<td>0.19</td>
<td>2.e−3</td>
<td>33.86</td>
</tr>
<tr>
<td>0.15</td>
<td>1.e−3</td>
<td>27.34</td>
</tr>
<tr>
<td>0.12</td>
<td>6.e−4</td>
<td>23.27</td>
</tr>
<tr>
<td>0.10</td>
<td>1.e−4</td>
<td>21.99</td>
</tr>
<tr>
<td>0.09</td>
<td>9.e−5</td>
<td>20.09</td>
</tr>
</tbody>
</table>

Fig. 1. Estimated convergence orders for nonstructured uniformly regular meshes.

To estimate the convergence order, we compute the following expression for two values of $h$:

$$r(1/h_1, 1/h_2) = \frac{\log(e(h_2)/e(h_1))}{\log(h_2/h_1)},$$

(93)
Table 2  Test 2. Errors for nonuniformly regular meshes

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.888577</td>
<td>0.1537810 0.4384070 2.99e−2</td>
<td>7.13e−2 5.53e−3 9.67e−3</td>
<td>0.403898</td>
<td>0.0203420 0.0927342 2.44e−3</td>
<td>6.62e−3 1.33e−4 3.97e−4</td>
<td>0.261346</td>
</tr>
</tbody>
</table>

where $e(h)$ is the norm of the difference between the exact and approximated solution ($H^1$-norm for the velocity and $L^2$-norm for the pressure).

We use unstructured uniformly regular meshes with $5 \times 5$, $15 \times 15$, $35 \times 35$, $55 \times 55$ and $75 \times 75$ nodes, and triangular finite elements of degree 2, 3 and 4 for velocity and pressure. Figure 1 shows the slope of the error curves. We obtain an excellent agreement with the theoretical predictions.

Table 3  Test 2. Estimated convergence order for nonuniformly regular meshes

<table>
<thead>
<tr>
<th>r(h)</th>
<th>Velocity degree 2</th>
<th>Pressure degree 2</th>
<th>Velocity degree 3</th>
<th>Pressure degree 3</th>
<th>Velocity degree 4</th>
<th>Pressure degree 4</th>
</tr>
</thead>
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<tr>
<td>0.888577, 0.403898</td>
<td>1.9701855</td>
<td>2.5655639</td>
<td>3.0157996</td>
<td>3.1755816</td>
<td>4.0510126</td>
<td>4.7312248</td>
</tr>
<tr>
<td>0.403898, 0.261346</td>
<td>2.1034308</td>
<td>2.4213809</td>
<td>3.1205766</td>
<td>3.0397084</td>
<td>4.1324295</td>
<td>4.6341359</td>
</tr>
<tr>
<td>0.261346, 0.193169</td>
<td>2.0853611</td>
<td>2.3892076</td>
<td>3.0892466</td>
<td>3.0269722</td>
<td>4.1032615</td>
<td>4.6401100</td>
</tr>
<tr>
<td>0.193169, 0.153203</td>
<td>2.0976119</td>
<td>2.2264732</td>
<td>3.0885004</td>
<td>3.0546166</td>
<td>4.0908042</td>
<td>4.4741609</td>
</tr>
<tr>
<td>0.153203, 0.126940</td>
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<td>2.4295187</td>
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<td>3.1612073</td>
<td>4.2143989</td>
<td>4.6965576</td>
</tr>
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<td>2.0595400</td>
<td>2.1205955</td>
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</tr>
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<td>2.3032472</td>
<td>3.1911315</td>
<td>3.1720994</td>
<td>4.1955161</td>
<td>4.5568381</td>
</tr>
</tbody>
</table>

6.2  Test 2: Nonuniformly regular meshes

We have solved the same problem as in Test 1 with nonuniformly regular meshes satisfying Hypothesis 2.2. In this case, we set $h$ as

$$h = \frac{1}{N} \sum_{k \in T_h} h_k,$$

where $N$ = number of elements in $T_h$. As in Test 1 we use finite elements of degree 2, 3 and 4. To estimate the convergence order we use expression (93). The fit to theoretical orders is as good as for uniformly regular meshes but we observe slightly less accuracy (see Tables 2 and 3).

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REFERENCES


