Cubical Cohomology Ring of 3D Pictures

Rocio Gonzalez-Diaz, Maria Jose Jimenez, and Belen Medrano

Abstract. Cohomology and cohomology ring of three-dimensional (3D) objects are topological invariants that characterize holes and their relations. Cohomology ring has been traditionally computed on simplicial complexes. Nevertheless, cubical complexes deal directly with the voxels in 3D images, no additional triangulation is necessary, facilitating efficient algorithms for the computation of topological invariants in the image context. In this paper, we present formulas to directly compute the cohomology ring of 3D cubical complexes without making use of any additional triangulation. Starting from a cubical complex \( Q \) that represents a 3D binary-valued digital picture whose foreground has one connected component, we compute first the cohomological information on the boundary of the object, \( \partial Q \) by an incremental technique; then, using a face reduction algorithm, we compute it on the whole object; finally, applying the mentioned formulas, the cohomology ring is computed from such information.

Key words: Cohomology ring; cubical complexes; 3D digital images.

1 Introduction

Many computer application areas involve topological methods which usually mean a significant reduction in the amount of data. Homology is an algorithmically computable topological invariant that characterizes an object by its “holes” (in any dimension). Informally, holes of a 3D-object are its connected components in dim. 0, its tunnels in dim. 1 and its cavities in dim. 2. Cohomology is a topological invariant obtained by an algebraic duality of the notion of homology. Although the formal definition of cohomology is motivated primarily by algebraic considerations, homology and cohomology of 3D objects are isomorphic, that is, they provide the same topological information. Nevertheless, cohomology has an additional ring structure provided by the cup product (denoted by \( \smile \)). The cup product can be seen as the way the holes obtained in homology are related to each other. For example, think of the torus, and the wedge sum of two loops and a 2-sphere. Both objects have two tunnels and one cavity; but the cavity (\( \gamma \)) of the first object can be decomposed in the product of the two

\* Partially supported by Junta de Andalucía (FQM-296 and TIC-02268).
tunnels ($\alpha$ and $\beta$), that is, $\alpha \sim \beta = \gamma$; the cavity of the second object cannot (see Fig. 1). This information would contribute to a better understanding of the degree of topological complexity of the analyzed digital object, and would shed light on its geometric features.

In [4, 5], a method for computing the cohomology ring of 3D binary digital images is stated. In those works, the cohomology ring computation is performed over the (unique) simplicial complex associated to the digital binary-valued picture using the 14-adjacency. However, one could assert that a more natural combinatorial structure when dealing with 3D digital images is the one provided by cubical complexes. One way to compute the cohomology ring of a cubical complex $Q$ is to convert it into a simplicial complex $K_Q$ by subdividing each cell and applying the known formulas for computing the cohomology ring of $K_Q$. In this paper, we present formulas to directly compute the cohomology ring of 3D cubical complexes without making use of additional triangulations. Besides, we describe a strategy to tackle the cohomology ring computation on a 3D binary-valued digital picture. This paper extends a preliminary version (see [2]).

The paper is organized as follows. In Section 2, we recall the concept of AT-model and extend it to general polyhedral cell complexes; given the AT-model for a polyhedral cell complex, we give the formulas of a new AT-model obtained after a subdivision; this result will be the key to prove the validity of the formulas of the cohomology ring of cubical complexes. In Section 3, formulas for computing the cohomology ring of 3D cubical complexes are established. Section 4 is devoted to describe the process to obtain an AT-model of a 3D digital image that provides the ingredients for the cohomology ring computation. Finally, some conclusions and plans for future are drawn in Section 5.

![Fig. 1. On the left, a hollow torus and its two tunnels. On the right, the wedge sum of a 2-sphere and two loops.](image)

2 AT-models for Polyhedral Cell Complexes

In this section, we recall first the concept of AT-model (see [4, 5]) for a cell complex, which consists of an algebraic set of data that provides homological information. Given the AT-model for a polyhedral cell complex, we show the
formulas of a new AT-model obtained after a subdivision; this result will be the key for the formulas of the cohomology ring of cubical complexes in the next section.

Since we are working with objects embedded in $\mathbb{R}^3$, the homology groups are torsion-free [1, ch.10], so computing homology over a field is enough to characterize shapes [11, p. 332]. This fact, together with the isomorphism (over any field) between the homology and cohomology groups [11, p. 320], enables us to consider $\mathbb{Z}/2$ as the ground ring throughout the paper.

A polyhedral cell complex $P$ in $\mathbb{R}^3$, is given by a finite collection of cells which are convex polytopes (vertices, edges, polygons and polyhedra), together with all their faces and such that the intersection between two of them is either empty or a face of each of them. A proper face of $\sigma \in P$ is a face of $\sigma$ whose dimension is strictly less than the one of $\sigma$. A facet of $\sigma$ is a proper face of $\sigma$ of maximal dimension. A maximal cell of $P$ is a cell of $P$ which is not a proper face of any other cell of $P$. Observe that if the cells of $P$ are $n$-simplices, $P$ is a simplicial complex (see [11]); in the case that the cells of $P$ are $n$-cubes, then $P$ is a cubical complex (see [9]). A $q$-cell of either a simplicial complex or a cubical complex can be denoted by the list of its vertices.

For any graded set $S = \{S_q\}_q$, one can consider formal sums of elements of $S_q$, which are called $q$-chains, and which form abelian groups with respect to the component-wise addition (mod 2). These groups are called $q$-chain groups and denoted by $C_q(S)$. The collection of all the chain groups associated to $S$ is denoted by $C(S) = \{C_q(S)\}_q$ and called also chain group, for simplicity. Let $\{s_1, \ldots, s_m\}$ be the elements of $S_q$ for a fixed $q$. Given two $q$-chains $c_1 = \sum_{i=1}^{m} \alpha_i s_i$ and $c_2 = \sum_{i=1}^{m} \beta_i s_i$, where $\alpha_i, \beta_i \in \mathbb{Z}/2$ for $i = 1, \ldots, m$, the expression $\langle c_1, c_2 \rangle$ refers to $\sum_{i=1}^{m} \alpha_i \cdot \beta_i \in \mathbb{Z}/2$. For example, fixed $i$ and $j$, the expression $\langle c_i, s_j \rangle$ is $\alpha_i$ and $\langle s_i, s_j \rangle$ is 1 if $i = j$ and 0 otherwise.

The polyhedral chain complex associated to the polyhedral cell complex $P$ is the collection $C(P) = \{C_q(P), \partial_q\}_q$ where:

(a) each $C_q(P)$ is the corresponding chain group generated by the $q$-cells of $P$;
(b) the boundary operator $\partial_q : C_q(P) \to C_{q-1}(P)$ connects two immediate dimensions. The boundary of a $q$-cell is the formal sum of all its facets. It is extended to $q$-chains by linearity.

For example, consider a triangle $(v_1, v_j, v_k)$ with vertices $v_1, v_j, v_k$. The boundary of the triangle is the formal sum of its edges, that is, $\partial_2(v_j, v_k) = (v_1, v_j) + (v_j, v_k) + (v_k, v_1)$.

Given a polyhedral cell complex $P$, an algebraic-topological model (AT-model) for $P$ is a set of data $(P, H, f, g, \phi)$, where $H$ is a graded subset of $P$ and $f, g, \phi$ are three families of maps $\{f_q : C_q(P) \to C_q(H)\}_q$, $\{g_q : C_q(H) \to C_q(P)\}_q$ and $\{\phi_q : C_q(P) \to C_{q+1}(P)\}_q$, such that, for each $q$:

1. $f_q g_q = \text{id}_{C_q(H)}$, $\phi_{q-1} \partial_q + \partial_{q+1} \phi_q = \text{id}_{C_q(P)} + g_q f_q$, $f_q-1 \partial_q = 0$, $\partial_q g_q = 0$;
2. $\phi_{q+1} \phi_q = 0$, $f_{q+1} \phi_q = 0$, $\phi_q g_q = 0$.

As a result, the chain group $C(H)$ is isomorphic to the homology (and to the cohomology) of $P$. In particular, the number of vertices of $H$ coincides with
the number of connected components of \( P \), the number of edges of \( H \) with the number of tunnels of \( P \) and the number of 2-cells of \( H \) with the number of cavities of \( P \). Fixed \( q \), for each \( \sigma \in H_q \), \( g_q(\sigma) \) is a representative cycle of a homology generator of dim. \( q \). Define an homomorphism \( \sigma^* f_q : C_q(P) \to \mathbb{Z}/2 \) such that if \( \mu \) is a \( q \)-cell of \( P \) then

\[
\sigma^* f_q(\mu) := \langle \sigma, f_q(\mu) \rangle \mod 2.
\]

Then, \( \sigma^* f_q \) is a representative cocycle of a cohomology generator of dim. \( q \). An isomorphism between \( C(H) \) and the homology (resp. cohomology) of \( P \) maps each \( \sigma \in H \) to the homology class represented by \( g_q(\sigma) \) (resp. the cohomology class represented by \( \sigma^* f_q \)) (see [4, 5]).

![Fig. 2.](image)

a) An abstract cubical representation \( Q \) of the hollow torus; b) the paths \( \gamma(v_i, v_0) \); c) the “path” \( c_e \), for any edge \( e \) of \( Q \).

From now on, we will omit subscripts on behalf of simplicity.

**Example 1.** Let \( Q \) be an abstract cubical representation of the hollow torus given in Fig. 2. An AT-model for \( Q \) is the set of data \((Q, H, f, g, \phi)\) given in the following table:

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( f )</th>
<th>( \phi )</th>
<th>( H )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_0 )</td>
<td>( v_0 )</td>
<td>0</td>
<td>( v_0 )</td>
<td>( v_0 )</td>
</tr>
<tr>
<td>( v_1, i = 1, \ldots, 8 )</td>
<td>( v_0 )</td>
<td>( \gamma(v_i, v_0) )</td>
<td>( v_0 )</td>
<td>( v_0 )</td>
</tr>
<tr>
<td>( a_i, i = 1, 2 )</td>
<td>( b_i )</td>
<td>( e_{a_i} )</td>
<td>( b_i )</td>
<td>( \alpha_i )</td>
</tr>
<tr>
<td>( b_i, i = 1, 2 )</td>
<td>( b_i )</td>
<td>0</td>
<td>( b_i )</td>
<td>( \alpha_i )</td>
</tr>
<tr>
<td>any edge ( b \neq a_i, b_i )</td>
<td>0</td>
<td>( c_b )</td>
<td>( c_b )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>any square ( \sigma \neq c )</td>
<td>( c )</td>
<td>0</td>
<td>( c )</td>
<td>( \beta )</td>
</tr>
</tbody>
</table>

where \( a_1 \in \{ (v_3, v_0), (v_4, v_8) \} \); \( a_2 \in \{ (v_1, v_7), (v_2, v_6) \} \); \( b_1 = (v_0, v_2) \); \( b_2 = (v_0, v_4) \); \( c = (v_0, v_2, v_4, v_8) \); \( \gamma(v_3, v_0) \) is the only path in \( Q \) from \( v_1 \) to \( v_0 \) in Fig. 2.b (for example, \( \gamma(v_3, v_0) = (v_5, v_7) + (v_1, v_5) + (v_0, v_1) \)); given an edge \( e \) of \( Q \), \( c_e \) is the sum of the squares that correspond to the path starting from \( e \) and following the arrows in Fig. 2.c (for example, \( c_{(v_3, v_1, v_7)} = (v_0, v_1, v_7) + (v_3, v_5, v_4, v_7) + \).
(v₀, v₁, v₂, v₃, v₄)); representative cycles of homology generators are the vertex v₀, the tunnels α₁ = (v₀, v₁) + (v₁, v₂) + (v₂, v₀) and α₂ = (v₀, v₃) + (v₃, v₄) + (v₄, v₀) and the cavity β which is the sum of the 9 squares of Q.

An algorithm for computing AT-models for polyhedral cell complexes appears for example in [4, 5]. In fact, in those papers, the algorithm is designed for simplicial complexes but the adaptation to polyhedral cell complexes is straightforward. That algorithm runs in time $O(m^3)$ where $m$ is the number of cells of the given polyhedral cell complex. If an AT-model $(P, H, f, g, ϕ)$ for a polyhedral cell complex $P$ is computed using that algorithm then it is also satisfied that if $a ∈ H$ then $f(a) = a$ and $a ∈ g(a)$.

Let $P$ be a polyhedral cell complex. Fixed $g$, we say that a $q$-cell $α ∈ P$ is subdivided into two new $q$-cells $α₁ ⊈ P$ and $α₂ ⊈ P$ by a new $(q - 1)$-cell $e ⊈ P$ (see, for example, Fig. 3) if:

(a) $e$ is a facet of $α₁$ and $α₂$;
(b) $α₁ ∪ α₂ = α$;
(c) $α₁ \cap α₂ = e$.

The following lemma establishes how to obtain a new AT-model for $P$ after subdividing by a $(q - 1)$-cell $e$ a $q$-cell $α$ of $P$ into two new $q$-cells $α₁$ and $α₂$. This result will be used to prove the equivalence between the new formula to compute the cup product on cubical complexes and the classical one on simplicial complexes.

**Lemma 1.** Let $(P, H, f, g, ϕ)$ be an AT-model for a polyhedral cell complex $P$ computed using the algorithm given in [4, 5]. Let $α$ be a $q$-cell, which is subdivided into two $q$-cells $α₁$ and $α₂$ by a new $(q - 1)$-cell $e$. Let $H' := (H \setminus \{α\}) \cup \{α₁\}$ if $α ∈ H$, and $H' := H$ otherwise; $P' := (P \setminus \{α\}) \cup \{α₁, α₂, e\}$; and $ϕ'$ the boundary operator of $P'$ given by: $ϕ'(c) := ϕ(c) + (α, ϕ(c))(α₁ + α₂)$ for any $c ∈ P' \setminus \{e, α₁, α₂\}$. Denote $ϕ(α₁) + e$ by $A$, and $ϕ(α₂) + e$ by $B$. Then, the set $(P', H', f', g', ϕ')$ is an AT-model for $P'$, where $f'$, $g'$ and $ϕ'$ are given by:

- $f'(α₁) := f(α) + (α, f(α))(α₁ + α)$; $f'(α₂) := 0$; $f'(e) := f(A) + f(B)$;
- $f'(σ) := f(σ) + (α, f(σ))(α₁ + α)$, for any $σ ∈ P' \setminus \{α₁, α₂, e\}$;
- $ϕ'(α₁) := ϕ(α)$; $ϕ'(α₂) := 0$; $ϕ'(e) := α₂ + ϕ(B) + (α, ϕ(B))(α₁ + α₂)$;
- $ϕ'(σ) := ϕ(σ) + (α, ϕ(σ))(α₁ + α₂)$, for any $σ ∈ P' \setminus \{α₁, α₂, e\}$;
– If $\alpha \in H$, $g'(\alpha_1) := g(\alpha) + \alpha + \alpha_1 + \alpha_2$;
$g'(\gamma) := g(\gamma) + \langle \alpha, g(\gamma) \rangle \alpha + \alpha_1 + \alpha_2$, for any $\gamma \in H' \setminus \{\alpha_1\}$.

Proof. We have to check that $(P', H', f', g', \phi')$ is an AT-model for $P'$. We will only check that $f'g' = id$ and $id + gf' = \phi \partial' + \partial' \phi'$. The rest of the conditions are left to the reader.

Let $\gamma \in H'$, $\gamma \neq \alpha_1$, and $\sigma \in P' \setminus \{\alpha_1, \alpha_2, e\}$.

– $f'g' = id$:

$f'g'(\gamma) = f'(g(\gamma) + \langle \alpha, g(\gamma) \rangle \alpha + \langle \alpha, g(\gamma) \rangle f(\alpha) + \langle \alpha, g(\gamma) \rangle g(\alpha)) = g(\gamma) + \langle \alpha, g(\gamma) \rangle f(\alpha) + \langle \alpha, g(\gamma) \rangle g(\alpha) + \langle \alpha, g(\gamma) \rangle (f(\alpha) + g(\alpha)) (\alpha + \alpha_1) = \gamma$.

If $\alpha \in H$: $f'g'(\alpha_1) = f'(g(\alpha)) f(\alpha) + f'(g(\alpha)) g(\alpha) + (\alpha, f(\alpha)) (\alpha + \alpha_1) = \alpha_1$.

– $id + gf' = \phi \partial' + \partial' \phi'$:

$\alpha_1 + gf'(\alpha_1) = \alpha_1 + g(f(\alpha) + \langle \alpha, f(\alpha) \rangle \alpha + \langle \alpha, f(\alpha) \rangle g(\alpha)) = \alpha_1 + g(f(\alpha) + \langle \alpha, f(\alpha) \rangle \alpha + \langle \alpha, f(\alpha) \rangle g(\alpha) + \langle \alpha, f(\alpha) \rangle g(\alpha)) (\alpha + \alpha_1 + \alpha_2) = \alpha_1 + \alpha + \phi(A)$.

Besides, $\sigma + g'f'(\sigma) = g'(f(\sigma) + \langle \alpha, f(\sigma) \rangle \alpha + \langle \alpha, f(\sigma) \rangle g'(\alpha_1) = \sigma + g(f(\sigma) + \langle \alpha, f(\sigma) \rangle \alpha + \langle \alpha, g(f(\sigma)) \alpha + \langle \alpha, f(\sigma) \rangle f(\alpha) + \langle \alpha, g(f(\sigma)) \alpha + \langle \alpha, f(\sigma) \rangle g(\alpha) + \alpha + \alpha_1 + \alpha_2 = \sigma + g(f(\sigma) + \langle \alpha, g(f(\sigma)) \alpha + \alpha + \alpha_1 + \alpha_2$.

Observe that $h : C(H) \to C(H')$, given by $h(\alpha) = 1$ if $\alpha \in H$ and $h(\sigma) = \sigma$ for any $\sigma \in H \setminus \{\alpha_1\}$, is a chain-group isomorphism.

3 3D Cubical Cohomology Ring

In [12, 9], the authors consider cubical complexes as the geometric building blocks to compute the homology of digital images. In this section, we adapt to the
Cubical Cohomology Ring of 3D Pictures

cubical setting, the method developed in [4, 5] for computing the simplicial co-

homology ring of 3D binary-valued digital pictures. We must mention [14, 10]
as related works dealing with the cup product on cubical chain complexes in a

theoretical context.

Since we are working with objects embedding in \( \mathbb{R}^3 \), it is satisfied that homol-
ygy and cohomology are isomorphic. However, cohomology has the advantage

over homology of having an additional ring structure given by the \text{cup product},

that is a topological invariant. This product provides information about the re-
lationship between the generators of (co)homology, that enables to discriminate,

for instance, pairs of cycles in different contexts, as in Fig. 1. Notice that, in

3D, the only non-trivial cup products are those corresponding to elements of

cohomology of dim. 1. If the cup product of two elements of cohomology of dim.

1 is not zero, then it is a sum of elements of cohomology of dim. 2. Recall that
given an AT-model \((P, H, f, g, \phi)\) for a polyhedral cell complex \(P\), it is satisfied

that \(H\) is isomorphic to the homology and to the cohomology of \(P\).

3.1 Cohomology Ring of Simplicial Complexes

We recall now how the cup product is defined in the simplicial setting using

AT-models:

\[
\text{Definition 1. \cite{4, 5} Let } K \text{ be a simplicial complex. It is assumed that the ver-
tices of } K \text{ are ordered. Let } (K, H, f, g, \phi) \text{ be an AT-model for } K. \text{ Let } \{\beta_1, \ldots, \beta_q\}
\text{ be the set of 2-simplices of } H \text{ and let } \alpha_1 \text{ and } \alpha_2 \text{ be two edges of } H. \text{ The cup}
\text{ product of } \alpha_1 \text{ and } \alpha_2 \text{ is:}
\sum_{k=1}^{q} ((\alpha_1 f \sim \alpha_2 f)(g(\beta_k))) \mod 2;
\]

where \(\alpha_1 f \sim \alpha_2 f\) on a 2-simplex \((v_i, v_j, v_k)\) with vertices \(v_i < v_j < v_k\) is

\((\alpha_1, f(v_i, v_j)) \cdot (\alpha_2, f(v_j, v_k)); \text{ and } (\alpha_1 f \sim \alpha_2 f) \text{ is extended to 2-chains (sums}

of 2-simplices) by linearity.\)

Observe that for each \(k\), \(g(\beta_k)\) is a sum of 2-simplices representing one cavity.

Then, \((\alpha_1 f \sim \alpha_2 f)(g(\beta_k))\) is a sum of 0s and 1s over \(\mathbb{Z}/2\) whose result is 0 or 1.

Therefore, \(\sum_{k=1}^{q} ((\alpha_1 f \sim \alpha_2 f)(g(\beta_k)))/2_k\) is a sum of 2-simplices of \(H\) of dim. 2,

representing the cavities obtained by “multiplying” the two representative cycles

\(g(\alpha_1)\) and \(g(\alpha_2)\) (think of the two tunnels of a hollow torus).

It is known that two objects with non-isomorphic cohomology rings, are not
topologically equivalent (more precisely, they are not homotopic) \cite{11}. To use the
information of the cohomology ring for this aim, one can construct both matrices

\(M\) and \(M'\) collecting the results of the cup product of cohomology classes of dim.

1 of each object, if the rank of \(M\) and \(M'\) are different, then we can assert that
both objects are not homotopic (see \cite{5}).
3.2 Cohomology Ring of Cubical Complexes

Now, let $Q$ be a cubical complex. Our aim is to obtain a direct formula for the cup product on $Q$ without making use of any triangulation.

Suppose that the vertices of $Q$ are labeled in a way that:

(P1) Each square $(v_i, v_j, v_k, v_\ell)$ of $Q$ with vertices $v_i < v_j < v_k < v_\ell$ has the edges $(v_i, v_j), (v_i, v_k), (v_j, v_\ell)$ and $(v_k, v_\ell)$ in its boundary.

For example, a cubical complex whose set of vertices is a subset of $\mathbb{Z}^3$ (the set of points with integer coordinates in 3D space $\mathbb{R}^3$) with vertices labeled using the lexicographical order, satisfies P1.

**Definition 2.** Let $Q$ be a cubical complex satisfying P1 and $(Q, H, f, g, \phi)$ an AT-model for $Q$. Let $\{\beta_1, \ldots, \beta_q\}$ be the set of squares of $H$ and let $\alpha_1$ and $\alpha_2$ be two edges of $H$. The cup product of $\alpha_1$ and $\alpha_2$ is

$$\sum_{k=1}^{q} ((\alpha_1 f \circ Q \alpha_2 f)(g(\beta_k))) \beta_k \mod 2;$$

where $\alpha_1 f \circ Q \alpha_2 f$ on a square $(v_i, v_j, v_k, v_\ell)$ with vertices $v_i < v_j < v_k < v_\ell$ (see Fig. 4) is:

$$\langle \alpha_1, f(v_i, v_j) \rangle \cdot \langle \alpha_2, f(v_j, v_\ell) \rangle + \langle \alpha_1, f(v_i, v_k) \rangle \cdot \langle \alpha_2, f(v_k, v_\ell) \rangle;$$

and $(\alpha_1 f \circ Q \alpha_2 f)$ is extended to 2-chains (sums of squares) by linearity.

For simplicity, we sometimes use the notations $(\alpha \circ Q \alpha')(\beta)$ for $(\alpha f \circ Q \alpha f')(g(\beta))$, and analogously for the simplicial cup product.

![Fig. 4. Scheme of the cubical cup product.](image)

**Example 2.** Let $Q$ be an abstract cubical representation of the hollow torus given in Fig. 2. Consider the AT-model $(Q, H, f, g, \phi)$ for $Q$, given in Example 1. Recall that $H = \{(v_0, v_0), (v_0, v_2), (v_0, v_4), (v_0, v_2, v_4, v_8)\}$; and $g(v_0) = v_0$, $g(v_0, v_2) = (v_0, v_1) + (v_1, v_2) + (v_2, v_0)$, $g(v_0, v_4) = (v_0, v_3) + (v_3, v_4) + (v_4, v_0)$ and $g(v_0, v_2, v_4, v_8)$ is the sum of the squares of $Q$, representing the connected component, the two tunnels and the cavity.
Apply the formula given in Def. 2 in order to obtain the cup product of $(v_0, v_2)$ and $(v_0, v_4)$ in $H$:

\[
(\langle \cd (v_0, v_2), f \rangle \cdot \langle \cd (v_0, v_4), f \rangle)
= 1 \cdot 1 + 0 \cdot 0 = 1.
\]

then, $(v_0, v_2) \sim_Q (v_0, v_4) = (v_0, v_2, v_4, v_8)$. Recall that $(v_0, v_2, v_4, v_8)$ is the square in $H$ representing the cavity of the hollow torus. Therefore, the product of the two tunnels of the hollow torus is the cavity.

The following theorem shows the validity of the definition of $\sim_Q$ (Def. 2).

That is, it is stated that we obtain the same result by applying the formula of Def. 2 to compute the cup product on the cubical complex, than making first a triangulation in order to obtain a simplicial complex, and applying the classical definition of the cup product given in Def. 1, afterwards.

Consider successive subdivisions of each cube of a given cubical complex $Q$ until each one is converted in six tetrahedra, and such that each square $(v_i, v_j, v_k, v_\ell)$ of $Q$ is subdivided by the edge $(v_i, v_\ell)$ (see Fig. 5.d). Let us denote this resulting simplicial complex by $K_Q$. Observe that with this particular subdivision, if $(v_p, v_q, v_r)$ is a 2-simplex of $K_Q$, with $v_p < v_q < v_r$, obtained by a subdivision of a square of $Q$, then $(v_p, v_q)$ and $(v_q, v_r)$ will correspond to edges in $Q$.

**Theorem 1.** Let $(Q, H, f, g, \phi)$ be an AT-model for $Q$. Let $\alpha$ and $\alpha'$ be two edges of $H$ and $\beta \in H$ a square. Let $(K_Q, H', f', g', \phi')$ be the AT-model for $K_Q$ obtained after successively applying Lemma 1. Then,

\[
(\alpha \sim_Q \alpha')(\beta) = (\alpha \sim \alpha')(h(\beta))
\]

where $\sim$ is the simplicial cup product given in Def. 1, $\sim_Q$ is the cubical cup product given in Def. 2, and $h : C(H) \to C(H')$ is the isomorphism defined at the end of Section 2.

![Fig. 5. From a) to c), successive subdivisions; d) a cube subdivided in 6 tetrahedra.](image-url)
Each point of $\mathbb{Z}_4$. From Digital Pictures to Cubical Complexes

cohomology ring is computed from such an AT-model.

Then, having in mind that the homology of $Q$ contains the homology of $\partial Q$, we obtain an AT-model for $Q$ with the representative cycles of homology generators lying in $\partial Q$; finally, applying the formula given in Section 3, the cohomology ring is computed from such an AT-model.

4 Cubical Cohomology Ring of 3D Digital Pictures

In this section, we develop the main bulk of the paper: beginning from a cubical complex $Q$ that represents a 3D binary digital picture whose foreground has one connected component, first we compute an AT-model for the boundary $\partial Q$ of the object; then, having in mind that the homology of $\partial Q$ contains the homology of $Q$, we obtain an AT-model for $Q$ with the representative cycles of homology generators lying in $\partial Q$; finally, applying the formula given in Section 3, the cohomology ring is computed from such an AT-model.

4.1 From Digital Pictures to Cubical Complexes

Each point of $\mathbb{Z}^3$ can be identified with a unit cube (called voxel) centered at this point, with facets parallel to the coordinate planes. This gives us an intuitive and simple correspondence between points in $\mathbb{Z}^3$ and voxels in $\mathbb{R}^3$.

Consider a 3D binary digital picture $I = (\mathbb{Z}^3, 26, 6, B)$, where $B$ (the foreground) is finite, having $\mathbb{Z}^3$ as the underlying grid and fixing the 26-adjacency for the points of $B$ and the 6-adjacency for the points of $\mathbb{Z}^3 \setminus B$ (the background). We say that a voxel $V$ is in the boundary of $I$ if $V \in B$ (i.e., if the point of $\mathbb{Z}^3$ identified with $V$ is in $B$) and $V$ has a 6-neighbor in $\mathbb{Z}^3 \setminus B$.

Take the cubical complex $Q$ for $B$ whose elements are the unit cubes (voxels) centered at the points of $B$ together with all their faces. Observe that this cubical complex with vertices labeled by the corresponding cartesian coordinates and considering the lexicographical order, satisfies (P1) (see page 8).

Without lack of generality, we consider that the foreground is connected.

The elements of $\partial Q$ are all the squares of $Q$ which are shared by a voxel of $B$ and a voxel of $\mathbb{Z}^3 \setminus B$ together with all their faces.
4.2 AT-model for $\partial Q$

Our interest now is to adapt the incremental algorithm for computing an AT-model given in [4, 5] to the particular complex $\partial Q$.

First, consider the set of edges and vertices of $\partial Q$ as a graph and compute a spanning forest $T$. Let $T_1, \ldots, T_m$ be the trees of $T$ corresponding to the connected components of $\partial Q$. Fixed $i, i = 1, \ldots, m$, take a vertex $v_i$ of $T_i$ and consider it as the root of $T_i$.

Algorithm 2 Computing an AT-model $(\partial Q, H, f, g, \phi)$ for $\partial Q$.

**INPUT:** The complex $\partial Q$,
the set $\{T_1, \ldots, T_m\}$ of trees of a spanning forest $T$ of $\partial Q$,
the set $\{v_1, \ldots, v_m\}$ of roots of the trees of $T$.

1. Initialize $f(\sigma) := \sigma, \phi(\sigma) := 0$ for any $\sigma \in \partial Q$; $H := \{v_1, \ldots, v_m\}$,
$U := \{v : v \text{ is a vertex of } \partial Q\}$,
$f(v) := v_i$ if $v$ is a vertex of $T_i$ for some $i, i = 1, \ldots, m$.
For $i = 1$ to $m$ do
   From $\ell = 1$ to the height of $T_i$ do
      For each vertex $v$ at level $\ell$, and edge $a$ linking $v$ with its parent $w$ do
         $\phi_i(v) := a + \phi_i(w), U := U \cup \{a\}, f(a) := 0$.
2. While there are edges in $\partial Q \setminus U$ do
   If there is a square $c \in \partial Q \setminus U$ with exactly one edge $a \in \partial Q \setminus U$ in its boundary do $U := U \cup \{a\}$,
   $f(a) := f(\partial(c) + a), \phi(a) := c + \phi(\partial(c) + a), f(c) := 0$.
   Else take an edge $a \in \partial Q \setminus U$ then $H := H \cup \{a\}$, $U := U \cup \{a\}$.
3. While there is a square $c$ in $\partial Q \setminus U$ do $U := U \cup \{c\}$.
   If $f(\partial(c)) = 0$ then $H := H \cup \{c\}$.
   Else take an edge $a$ in $f(\partial(c)$ then $H := H \setminus \{a\}$.
   For each edge $b$ in $\partial Q \setminus T$ do $f(b) := f(b) + \langle a, f(b) \rangle f(\partial(c))$
   $\phi(b) := \phi(b) + \langle a, f(b) \rangle (c + \phi(\partial(c)), f(c) := 0$.
4. For each $\sigma \in H$ do $g(\sigma) := \sigma + \phi(\sigma)$.

**OUTPUT:** the AT-model $(\partial Q, H, f, g, \phi)$ for $\partial Q$.

The auxiliary set $U$ is defined to indicate the cells which have already been used. In Step 1, neither the vertices nor the edges of $T_i$ create cycles except for the root $v_i$. In Step 3, if a square has edges in its boundary that created cycles in a previous step, then one of these cycles is destroyed. Otherwise, this square creates a new cycle (a cavity). In the last step, the representative cycles of homology generators are computed.

Observe that all the steps of Alg. 2 are quadratic in the number of elements of $\partial Q$ (worse case complexity) except for the last part of Step 3 which is cubic in the number of edges of $\partial Q \setminus T$.

**Example 3.** The AT-model $(\partial Q, H, f, g, \phi)$ of a hollow cube $\partial Q$ (see Fig. 7) is:

<table>
<thead>
<tr>
<th>Step</th>
<th>$\partial Q$</th>
<th>$f$</th>
<th>$\phi$</th>
<th>$H$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>$v$</td>
<td>$v$</td>
<td>$0$</td>
<td>$v$</td>
<td>$v$</td>
</tr>
<tr>
<td>$v_i$, $i = 1, \ldots, 7$</td>
<td>$\gamma(v_i, v)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Step 2</td>
<td>Any edge $b \in \partial Q \setminus T$</td>
<td>$0$</td>
<td>$c_b$</td>
<td>$0$</td>
<td>$c_b$</td>
</tr>
<tr>
<td>Step 3</td>
<td>$(v, v_2, v_4, v_6)$</td>
<td>$(v, v_2, v_4, v_6)$</td>
<td>$0$</td>
<td>$(v, v_2, v_4, v_6)$</td>
<td>$C$</td>
</tr>
</tbody>
</table>

where $\gamma(v_i, v)$ is the only path in $T$ from $v_i$ to $v$; $c_b$ is the square at which the arrow corresponding to an edge $b$ in Fig. 7.c points, except for $c_{(v_4, v_6)}$ which is $(v_4, v_5, v_6, v_7) + (v_1, v_3, v_5, v_7)$; and $C$ is the sum of the six squares of $\partial Q$.

![Fig. 7. a) A hollow cube $\partial Q$; b) a spanning tree $T$ with root $v$; c) the “paths” $c_b$; d) the cells of $H$.](image)

### 4.3 AT-model for $Q$

Now, we use a face reduction technique (see, for example, [3, 9, 13]) in order to obtain a cell complex $K$ such that homology, cohomology and cohomology ring of $K$ coincide with that of $Q$, and such that the cells of $\partial Q$ are also cells of $K$.

**Algorithm 3** Face Reduction Process.
Cubical Cohomology Ring of 3D Pictures

Input: A cubical complex \( Q \). Initially, \( K := Q \).
While there exist \( \sigma, \sigma' \in Q \setminus \partial Q \) such that \( \sigma' \) is in \( \partial(\sigma) \) do
  For each cell \( c \in K \) such that \( \sigma' \) is in \( \partial(c) \) do
    redefine \( \partial(c) \) as \( \partial(c + \sigma) \).
Remove \( \sigma \) and \( \sigma' \) from the current \( K \);
Output: the cell complex \( K \).

Fig. 8. a) A cubical complex \( Q \) composed by two cubes \( c \) and \( \sigma \) sharing a square \( \sigma' \) (in red) and all their faces; b) the squares of \( \partial(c) := \partial(c + \sigma) \) in \( Q' \).

Observe that after the face reduction process, we obtain a cell complex \( K \) with the same topological information as \( Q \) but with much less cells. Now, starting from an AT model for \( \partial Q \), compute an AT-model for \( K \) adding the cells of \( K \setminus \partial Q \) incrementally as follows:

Algorithm 4 AT-model for \( K \).

Input: An AT-model for \( \partial Q \): \((\partial Q, H, f, g, \phi)\) and the cells \( \{\sigma_1, \ldots, \sigma_m\} \) of \( K \setminus \partial Q \) ordered by increasing dimension. Initially, \( f_K(\sigma) := f(\sigma) \), \( \phi_K(\sigma) := \phi(\sigma) \), \( g_K(\sigma) := g(\sigma) \) for each \( \sigma \in \partial Q \); \( f_K(\sigma) := 0 \), \( \phi_K(\sigma) := 0 \), for each \( \sigma \in K \setminus \partial Q \); \( H_K := H \).
For \( i = 1 \) to \( i = m \) do
  take a cell, \( \sigma_i \), of \( f_K(\partial(\sigma_i)) \), then
  \( H_K := H \setminus \{\sigma_i\} \),
  For \( k = 1 \) to \( k = i - 1 \) do
    \( f_K(\sigma_k) := f_K(\sigma_k) + (\sigma, f_K(\sigma_k))f_K(\partial(\sigma_i)) \)
    \( \phi_K(\sigma_k) := \phi_K(\sigma_k) + (\sigma, f_K(\sigma_k))\phi_K(\partial(\sigma_i)) \)
Output: the AT-model \((K, H_K, f_K, g_K, \phi_K)\) for \( K \).

Observe that in the algorithm, a cycle is never created because the cycles of \( \partial Q \) are also cycles of \( K \). Therefore, when a cell \( \sigma_i \) of \( K \) is added, then \( f_K(\partial(\sigma_i)) \) is never null and therefore a class of homology is always eliminated (that is, a cell \( \sigma \) of \( f_K(\partial(\sigma_i)) \) is removed from \( H_K \)). Alg. 4 is \( \mathcal{O}(m^3) \), where \( m \) is the number of cells of \( K \setminus \partial Q \) (worst case complexity).
4.4 From Cubical Complexes to Digital Pictures

Now, given a digital picture $I$, suppose that we have computed an AT-model $(K, H_K, f_K, g_K, \phi_K)$ for $K$, following the steps given in Subsections 4.1, 4.2 and 4.3. For each $\sigma$ in $H_K$, $g_K(\sigma)$ is a representative cycle of a homology generator of $K$ and, therefore, of $Q$, since the homology of $K$ and $Q$ coincide and the representative cycles are in $\partial Q$. Recall that if $\sigma$ is a vertex, then $g_K(\sigma)$ is a vertex representing a connected component; if $\sigma$ is an edge, then $g_K(\sigma)$ is a sum of edges representing a tunnel; and if $\sigma$ is a square, then $g_K(\sigma)$ is a sum of squares representing a cavity.

Given a representative cycle $g_K(\sigma)$ of a homology generator, our aim in this subsection is to draw the equivalent cycle in the picture $I$.

First, if $g_K(\sigma)$ is a vertex then, $g_K(\sigma)$ is a face of a square in $\partial Q$. This square is shared by a voxel $V$ of $B$ and a voxel of $Z^3 \setminus B$. Then, associate the voxel $V$ to the vertex $g_K(\sigma)$.

Second, if $g_K(\sigma)$ is a sum of edges, suppose that $g_K(\sigma)$ is a simple cycle (if not, it can always be decomposed in simple ones). Visit all the edges of the cycle in order. If an edge, $a$, and the next edge, $b$, are facets of a square $\sigma \in \partial Q$, then associate the single voxel $V$ of $B$ which has $\sigma$ in its boundary, to the edges $a$ and $b$. After that, visit all the edges that have not been associated to any voxel. If a voxel $V$ is associated with the next edge of the current one, $a$, and there is a voxel $V' \in B$ having $a$ in its boundary, such that $V'$ is in the boundary of $I$ and $V$ and $V'$ are 6-neighbor, then associate $V'$ to $a$. If not, look at the previous edge and do the same procedure.

If not, take any voxel of $B$ that contains $a$, having a 6-neighbor voxel in $Z^3 \setminus B$.

Finally, if $g_K(\sigma)$ is a sum of squares, associate, to each square $\sigma$, the single voxel $V$ of $B$ which has $\sigma$ in its boundary.

4.5 Cohomology ring of $Q$

Given a digital picture $(Z^3, 26, 6, B)$, its associated cubical complex $Q$ and having computed an AT-model $(K, H_K, f_K, g_K, \phi_K)$ for $K$, the last step of the process...
is the computation of the cohomology ring. This can be performed using the
formula for the cubical cup product given in Def. 2.

*Example 4.* This example shows an application of $\sim_\partial$ to discriminate different
embeddings of the same object. Consider the cubical complex $Q_1$ (resp. $Q_2$) as-
related to a digital picture where the set $B$ consists in two once-linked “circles”
(resp. two unlinked “circles”). See Fig. 10. Both complexes have two tunnels and
no cavities, so these properties are not able to distinguish them.

Now, denote by $Q'_1$ and $Q'_2$, the cubical complexes associated to the back-
ground of $I_1$ and $I_2$ (white voxels of Fig. 10). Compute an AT-model for $Q'_i$, and
its cohomology ring, for $i = 1, 2$. We obtain that the multiplication table for the
cup product on $Q'_1$ is null whereas on $Q'_2$ is not (see Fig. 11). This fact allows
us to assert that the two complexes $Q'_1$ and $Q'_2$ are not topologically equivalent.

*Example 5.* Consider the picture in Fig. 12. The cubical complex associated to
the white voxels of the picture has 1 connected component, 6 tunnels and 3
cavities. The cup product is trivial for any two pairs of homology classes except
for two tunnels named as $\alpha_2$ and $\alpha_3$ wich is the sum of two of the three cavities
(see Fig. 12).

Finally, Consider the picture in Fig. 13. The cubical complex associated to
the white voxels of the picture has 1 connected component, 7 tunnels and 12
cavities. The results of the computation of the cup product can be seen in the
table of Fig. 13, where “CP $i \ j$" means the sum of the cavities $i$ and $j$.

5 Conclusions and Future Work

In this paper we present formulas to directly compute the cohomology ring of
3D cubical complexes and develop a method for the computation on 3D binary-
valued pictures. This computation on cubical complexes can be regarded as a
starting point to compute the cup product on general polyhedral cell complexes,
which is, in fact, our last goal. As related work, we must mention [8], where
homology of 3D pictures using particular cell structures provided by the 26-
adjacency is performed. The restriction to the 3D-world allows to work over
Fig. 11. In yellow, representative cycles of the two tunnels of $Q'_1$; in green, representative cycles of the two tunnels of $Q'_2$; at the bottom, the tables for the cup product on $Q'_1$ and $Q'_2$, where $\alpha_i$ (resp. $\alpha'_i$), $i = 1, 2$, are representative cycles of the two tunnels and $\beta_i$ (resp. $\beta'_i$), $i = 1, 2$, are representative cycles of the two cavities of $Q'_1$ (resp. $Q'_2$).

$\mathbb{Z}/2$, what facilitates the calculus. However, a harder task could be the one of extending the formulas of the cohomology ring to higher dimensions what could be applied to more general contexts out of digital images. In this sense, cohomology ring of nD simplicial complexes using AM-models and working in the integer domain, has been established in [6]. Another goal for future work is the one of applying theoretical results to irregular graph pyramids and compute the cohomology ring on the cell complexes associated to such structures. In the paper [7], representative cocycles for cohomology generators on irregular graph pyramids are computed, what can be considered a first step in this direction.

References

Fig. 12. A configuration of 6 linked circles and the results of the computation of the cup product of the cubical complex associated to the white voxels of the picture.
Fig. 13. A configuration of 7 linked circles and the results of the computation of the cup product of the cubical complex associated to the white voxels of the picture.