Rectifications of A_∞ -algebras ¹

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Abstract

In this paper, in the setting of connected DG-modules, we prove that, for any A_∞ -algebra $(M, m_1, m_2, m_3, \dots)$, there is a chain contraction from a DG-algebra onto the DG-module M such that the A_∞ -algebra structure induced by perturbation theory on M is the original one. In fact, the mentioned DG-algebra is a rectification of the A_∞ -algebra in the sense of Boardman and Vogt [BV73]. Appropriated dual results are given for A_∞ -coalgebras.

Key Words: A_∞ -algebra, A_∞ -coalgebra, contraction, tilde bar construction, tilde cobar construction

MSC: 18E30 18G35 16E45

1 Introduction

One classical result claims that each WP -algebra (strongly homotopy P -algebra) can be replaced by a strict P -algebra of the same homotopy type. More concretely, for each WP -algebra, X , there exists a strict P -algebra, MX , together with a strong deformation retraction $(MX, X, r : MX \rightarrow X, i : X \rightarrow MX)$ in which i is a homomorphism of WP -algebras [BV73]. Then, MX is called the *rectification* of X .

It seems to be a commonly accepted fact that the analogous statement holds also for A_∞ -algebras, that is, each A_∞ -algebra can be replaced by an associative DG-algebra of the same homotopy type. We have not been able, however, to find a proof of this statement in the literature.

The aim of this paper is to construct such a rectification of an A_∞ -algebra explicitly.

It is known that a chain contraction, which is a special type of chain homotopy equivalence, from a DGA-algebra A onto a DG-module M provides an A_∞ -algebra structure on M via the tensor trick [Gug77, GS86, GLS91]. In fact, the first transference of an A_∞ -algebra structure, in this sense, was made by Kadeishvili in [Kad80] for the case $M = H(A)$.

The main result in this paper can be seen as an inverse procedure of that one. In other words, we assert that, working in the context of connected DG-modules, for any

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A_∞ -algebra $(M, m_1, m_2, m_3, \dots)$, there exists a DG-algebra, A_M , and a chain contraction from A_M to M , such that the application of the tensor trick to this contraction yields the original A_∞ -algebra structure on M , defined by the family of morphisms $\{m_i\}$. In the dual case of A_∞ -coalgebras, our framework is the one of simply connected DG-modules.

In order to prove this result, we construct, by means of homological perturbation theory [HK91, GL89, GLS91], a chain contraction from the reduced cobar construction of the tilde bar construction of M , $\tilde{\Omega}\tilde{B}(M)$, to the DG-module M . This contraction represents a generalization (in the setting of A_∞ -algebras) of the result given by Munkholm in [Mun74] by which, for a DGA-algebra A , a contraction (called trivialized extension) from the DGA-algebra $\tilde{\Omega}\tilde{B}(A)$ to A is described.

This paper is organized as follows: in the next section we recall the preliminaries needed and state the notation used throughout the paper; the third section is devoted to our description of a family of morphisms that provide an A_∞ -(co)algebra structure via the tensor trick; in the forth section, we establish the main result of the paper by which any A_∞ -(co)algebra can be seen as a chain contraction from a (simply) connected DG-(co)algebra onto a DG-module.

2 Notations and preliminaries

Although relevant notions of homological algebra are recalled here, most of common concepts are not explicitly given. They might be consulted, for instance, in [CE56] and [Mac95].

Take a commutative ground ring, Λ , with unit. A *differential graded module* or DG-module, (M, d) , is a module M , graded on the non-negative integers, $M = \bigoplus_{n \geq 0} M_n$, endowed with a morphism of graded modules d of degree -1 such that, $d^2 = 0$. A DG-module M is called *connected* if $M_0 = \Lambda$ and *simply connected* if it is connected and $M_1 = 0$. Given a connected DG-module, M , the *reduced* module $\bar{M}$ is such that $\bar{M}_n = M_n$ for $n > 1$ and $\bar{M}_0 = 0$. In this paper, we shall always refer to connected DG-modules in the context of A_∞ -algebras and simply connected DG-modules when dealing with A_∞ -coalgebras.

We will denote the module $M \otimes \dots \otimes M$ by $M^{\otimes n}$, with $M^{\otimes 0} = \Lambda$; we will use the notation $f^{\otimes n}$ for the morphism

$$f \otimes \dots \otimes f : M^{\otimes n} \rightarrow N^{\otimes n}.$$

We will respect the Koszul convention for signs.

On the other hand, if $f : M^{\otimes i} \rightarrow N$ is a DG-module morphism and n is a non-negative integer, we will denote by $f^{[n]} : M^{\otimes n} \rightarrow N^{\otimes n-i+1}$ the morphism

$$f^{[n]} = \sum_{j=0}^{n-i} 1^{\otimes j} \otimes f \otimes 1^{\otimes n-i-j}.$$

The morphism $f^{[\cdot]} : \bigoplus_{j \geq i} M^{\otimes j} \rightarrow \bigoplus_{k \geq 1} N^{\otimes k}$ will be the one such that $f^{[\cdot]}|_{M^{\otimes n}} = f^{[n]}$.

If (M, d_M) is a DG-module, the *suspension* of M is defined as the DG-module (sM, d_{sM}) , where $(sM)_n = M_{n-1}$ and $d_{sM} = -d_M$. The *desuspension* of M is given by $(s^{-1}M)_n = M_{n+1}$

and differential $-d_M$ too. We will denote by $\uparrow$ and $\downarrow$ the suspension and desuspension morphisms, which shift the degree by $+1$ and -1 , respectively. A given morphism of graded modules of degree k , $f : M \rightarrow N$, induces $sf : sM \rightarrow sN$, given by $sf = (-1)^k \uparrow f \downarrow$. In fact, $d_{sM} = s d_M = - \uparrow d_M \downarrow$.

A *DG-algebra*, (A, d_A, μ_A) , is a DG-module endowed with an associative product, μ_A , compatible with the differential d_A and which has a unit $\eta_A : \Lambda \rightarrow A$, that is, $\mu_A(\eta_A \otimes 1) = \mu_A(1 \otimes \eta_A) = 1$. Sometimes we will use the notation $*_A$ for the product on A . If there is no confusion, subscripts will be omitted. A *DG-coalgebra* (C, d_C, Δ_C) is a DG-module provided with a compatible coproduct and counit $\xi_C : C \rightarrow \Lambda$ (so, $(\xi_C \otimes 1)\Delta_C = (1 \otimes \xi_C)\Delta_C = 1$).

Given a DG-module (M, d) , the *tensor module* of M is denoted by $T(M)$ and is constructed in the following way:

$$T(M) = \bigoplus_{n \geq 0} M^{\otimes n}.$$

The *tensor graduation* of $T(M)$, $| \cdot |_t$, is given by:

$$|a_1 \otimes \cdots \otimes a_n|_t = \sum_{i=1}^n |a_i|.$$

The differential structure in $T(M)$ is provided by the *tensor differential*, d_t , which is the morphism $d_M^{[\cdot]}$.

A product, μ , and a coproduct, Δ , can be naturally defined on $T(M)$, as follows:

- $\mu((a_1 \otimes \cdots \otimes a_n) \otimes (a_{n+1} \otimes \cdots \otimes a_{n+p})) = a_1 \otimes \cdots \otimes a_{n+p};$
- $\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^n (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n).$

Therefore, $T(M)$ acquires both structures of DG-algebra (denoted by $T^a(M)$) and DG-coalgebra ($T^c(M)$), though they are not compatible to each other (that is, $(T(M), \mu, \Delta)$ is not a Hopf algebra).

Every morphism of DG-modules $f : M \rightarrow N$ induces another one $T(f) : T(M) \rightarrow T(N)$, such that $T(f)|_{M^{\otimes n}} = f^{\otimes n}$.

The *reduced bar construction* of a connected DG-algebra A , $\bar{B}(A)$, is a DG-coalgebra whose module structure is given by

$$T(s\bar{A}) = \bigoplus_{n \geq 0} (s\bar{A})^{\otimes n}.$$

A typical element of $\bar{B}(A)$ is denoted by $\bar{a} = [a_1 | \cdots | a_n]$ and $[\] = 1 \in \Lambda$. The degree of a is given by the sum of the *tensor degree* $|a_1| + \cdots + |a_n|$ and the *simplicial degree* n , which appeals to the number of components or length of the element (also referred to as simplicial dimension).

The total differential $d_{\bar{B}}$ is given by the sum of the tensor differential, d_t (which is the natural one on the tensor product) and the *simplicial differential*, d_s (that depends on the product on A):

$$d_t = -(\uparrow d_A \downarrow)^{[\cdot]}; \quad d_s = (\uparrow \mu_A \downarrow^{\otimes 2})^{[\cdot]}.$$

The coproduct $\Delta_{\bar{B}} : \bar{B}(A) \rightarrow \bar{B}(A) \otimes \bar{B}(A)$ is the natural one on the tensor module.

Given a simply connected DG-coalgebra C , the *reduced cobar construction*, $\bar{\Omega}(C)$, is a DG-algebra whose underlying module is

$$T(s^{-1}\bar{C}) = \bigoplus_{n \geq 0} (s^{-1}\bar{C})^{\otimes n}.$$

A typical element of $\bar{\Omega}(C)$ will be written $\bar{c} = \langle c_1 | \cdots | c_n \rangle$, being $\langle \rangle = 1 \in \Lambda$. The total degree of $\bar{c}$ is $|\bar{c}|_{\bar{\Omega}} = |c_1| + \cdots + |c_n| - n$. We will refer to the length of the element, n , as the *cosimplicial degree* (or dimension).

The total differential $d_{\bar{\Omega}}$ is given by the sum of the tensor differential and the *cosimplicial differential* $d_{cos} = (\downarrow^{\otimes 2} \Delta_C \uparrow)^{[1]}$. The product on $\bar{\Omega}(C)$ is the natural one on the underlying module.

In the context of homological perturbation theory, the main input data are *chain contractions* (or simply, contractions) [EM53, GS86, LS87, GL89, HK91]: a contraction $c : \{N, M, f, g, \phi\}$ from a DG-module N to a DG-module M , consists in a particular homotopy equivalence determined by three morphisms f , g and ϕ ; being $f : N_{\star} \rightarrow M_{\star}$ (projection) and $g : M_{\star} \rightarrow N_{\star}$ (inclusion) two DG-module morphisms and $\phi : N_{\star} \rightarrow N_{\star+1}$ a homotopy operator, that is, $fg = 1_M$, and $\phi d_N + d_N \phi + gf = 1_N$. Moreover, these data are also required to satisfy:

$$f\phi = 0, \quad \phi g = 0, \quad \phi\phi = 0.$$

Notice that the homology of both DG-modules are isomorphic. We will also use the notation $(f, g, \phi) : N \Rightarrow M$ or simply $N \xRightarrow{c} M$ when confusion cannot arise.

Given a DG-module contraction,

$$c : \{N, M, f, g, \phi\}$$

we can establish the following ones [GL89, GLS91]:

- The *suspension contraction* of c , sc , which consists of the suspended DG-modules and the induced morphisms:

$$sc : \{sN, sM, sf, sg, s\phi\},$$

being $sf = \uparrow f \downarrow$, $sg = \uparrow g \downarrow$ and $s\phi = -\uparrow \phi \downarrow$, which are briefly expressed by f , g and $-\phi$.

- The *tensor module contraction*, $T(c)$, between the tensor modules of N and M :

$$T(c) : \{T(N), T(M), T(f), T(g), T(\phi)\},$$

where

$$T(\phi)|_{N^{\otimes n}} = \phi^{[\otimes n]} = \sum_{i=0}^{n-1} 1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes n-i-1}.$$

Now, we recall the concept of *perturbation datum*. Let N be a graded module and let $f : N \rightarrow N$ be a morphism of graded modules. The morphism f is defined to be *pointwise nilpotent* whenever for all $x \in N$, $x \neq 0$, there exists a positive integer n such that $f^n(x) = 0$. A *perturbation of a DG-module* N consists in a morphism of graded modules $\delta : N \rightarrow N$ of degree -1 , such that $(d_N + \delta)^2 = 0$. A *perturbation datum* of the contraction $c : \{N, M, f, g, \phi\}$ is a perturbation δ of the DG-module N satisfying that the composition $\phi \delta$ is pointwise nilpotent.

The main tool when dealing with contractions is the **Basic Perturbation Lemma** [Bro67, Gug72, LS87], which is an algorithm whose input is a contraction of DG-modules $c : \{N, M, f, g, \phi\}$ and a perturbation datum δ of c and whose output is a new contraction $c_\delta : \{(N, d_N + \delta), (M, d_M + d_\delta), f_\delta, g_\delta, \phi_\delta\}$ defined by the formulas

$$d_\delta = f \delta \Sigma_c^\delta g; \quad f_\delta = f (1 - \delta \Sigma_c^\delta \phi); \quad g_\delta = \Sigma_c^\delta g; \quad \phi_\delta = \Sigma_c^\delta \phi;$$

where $\Sigma_c^\delta = \sum_{i \geq 0} (-1)^i (\phi \delta)^i$.

The pointwise nilpotency of the composition $\phi \delta$ guarantees that the sums are finite for each particular element.

3 From contractions to A_∞ -structures

We find the origin of A_∞ -(co)algebras in [Sta63], where Stasheff set the concept of *strongly homotopy associativity* in the search of a homotopy invariant that plays the role of associativity.

We recall here the definition of A_∞ -algebra (resp. A_∞ -coalgebra) [Kad80, Pro84]. An A_∞ -algebra (respectively, A_∞ -coalgebra), is a DG-module (M, m_1) (resp., (M, Δ_1)) endowed with a family of morphisms of graded modules

$$m_i : M^{\otimes i} \rightarrow M \text{ (resp., } \Delta_i : M \rightarrow M^{\otimes i})$$

of degree $i - 2$ such that, for $i \geq 1$,

$$\sum_{n=1}^i \sum_{k=0}^{i-n} (-1)^{n+k+nk} m_{i-n+1}(1^{\otimes k} \otimes m_n \otimes 1^{\otimes i-n-k}) = 0, \quad (1)$$

$$\text{(resp., } \sum_{n=1}^i \sum_{k=0}^{i-n} (-1)^{n+k+nk} (1^{\otimes i-n-k} \otimes \Delta_n \otimes 1^{\otimes k}) \Delta_{i-n+1} = 0). \quad (2)$$

Starting from a contraction between a connected DG-algebra A and a DG-module M , the application of the tensor trick [Gug77, GS86, GLS91] provides a way of constructing a family of morphisms that makes the module inherit an A_∞ -algebra structure. Using this technique, in the following theorem, we will express these morphisms with regard to the component morphisms of the initial contraction. We draft a proof of the theorem with the only purpose of showing the tools used in the context of homological perturbation theory.

Theorem 3.1. [Kad80, GLS91] Let (A, d_A, μ) and (M, d_M) be a connected DG-algebra and a DG-module, respectively and $c : \{A, M, f, g, \phi\}$ a contraction between them. Then the DG-module M is endowed with an A_∞ -algebra structure by the morphisms

$$\begin{aligned} m_1 &= -d_M \\ m_n &= (-1)^{n+1} f \mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \dots \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}, \quad n \geq 2 \end{aligned}$$

where

$$\mu^{(k)} = \sum_{i=0}^{k-1} (-1)^{i+1} 1^{\otimes i} \otimes \mu_A \otimes 1^{\otimes k-i-1}.$$

Proof.

Starting from $c : \{A, M, f, g, \phi\}$, we can construct the contraction

$$T(sc) : \{T^c(s\bar{A}), T^c(s\bar{M}), Tf, Tg, T(-\phi)\}.$$

Now, in order to get the differential of the bar construction on the initial DG-module, we consider the simplicial differential as a perturbation datum of this contraction. We can easily check the pointwise nilpotency of $T(-\phi) d_s$ since $T(-\phi)$ does not affect to the simplicial dimension of the element, while d_s decreases this amount by one and so will be zero after a finite number of steps. Then, by applying the Basic Perturbation Lemma, a new contraction is obtained,

$$(\tilde{f}, \tilde{g}, \tilde{\phi}) : \tilde{B}(A) \Rightarrow (T^c(s\bar{M}), \tilde{d}),$$

where $(T^c(s\bar{M}), \tilde{d})$ is called the *tilde bar construction* of M [Sta63], denoted by $\tilde{B}(M)$ and the formula obtained for the perturbed differential is

$$\tilde{d} = d_t + \sum_{i \geq 0} (-1)^i Tf d_s (T(-\phi) d_s)^i Tg.$$

We call

$$\tilde{d}_1 = d_t, \quad \tilde{d}_i = (-1)^i Tf d_s (T(-\phi) d_s)^{i-2} Tg \quad \text{for } i \geq 2.$$

Let us consider the induced morphisms $m_n : M^{\otimes n} \rightarrow M$ of degree $n - 2$, with the following formulas:

$$m_n = (-1)^{[n/2]} \downarrow (\tilde{d}_n|_{(sM)^{\otimes n}}) \uparrow^{\otimes n},$$

where the brackets refer to the integer part. Then, it is easy to check that

$$m_n = (-1)^{n+1} f \mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \dots \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}.$$

Finally, the fact that $\tilde{d}^2 = 0$ can be translated into relations between the morphisms m_n which are, actually, the ones recalled in the formula 1.

□

An analogous theorem can be established for the case of an A_∞ -coalgebra structure, whose proof is completely dual to the one given above.

Theorem 3.2. [Kad80, GLS91] Let (C, d_C, Δ) and (M, d_M) be a simply connected DG-coalgebra and a DG-module, respectively, and $c : \{C, M, f, g, \phi\}$ a contraction between them. Then the DG-module M is endowed with an A_∞ -coalgebra structure by the morphisms

$$\begin{aligned}\Delta_1 &= -d_M \\ \Delta_n &= (-1)^{[n/2]+n+1} f^{\otimes n} \Delta^{(n)} \phi^{[\otimes n-1]} \Delta^{(n-1)} \dots \phi^{[\otimes 2]} \Delta^{(2)} g, \quad n \geq 2\end{aligned}$$

where

$$\Delta^{(k)} = \sum_{i=0}^{k-2} (-1)^i 1^{\otimes i} \otimes \Delta_C \otimes 1^{\otimes k-i-2}.$$

In this case, the simple connection of the DG-coalgebra guarantees that the formulas implied in the contraction $\bar{\Omega}(C) \Rightarrow \tilde{\Omega}(M)$, obtained by the Basic Perturbation Lemma, are finite.

Now we are concerned about the inverse process: given an A_∞ -(co)algebra, finding a suitable contraction that generates, in the sense given in this section, such a structure.

4 From A_∞ -structures to contractions

In this section, we explicitly construct rectification of an A_∞ -algebra. As a result, we establish that such a structure can be estructurely represented as a contraction from a DG-algebra onto a DG-module. Recall that by “DG-module” we shall mean a *connected* DG-module.

Munkholm obtained in [Mun74] a contraction between the reduced cobar construction of the reduced bar construction of a DGA-algebra A , $\bar{\Omega}\bar{B}A$, and the DGA-algebra itself.

$$c_A : \{\bar{\Omega}\bar{B}A, A, \alpha_A, \rho_A, h_A\}.$$

The same contraction, can be obtained by means of perturbation theory, for the connected case: starting from a “basic” contraction

$$\{\bar{\Omega}T^c(s\bar{A}), A, \alpha, \rho, h\},$$

and taking the perturbation datum $\delta = (\downarrow (-d_s) \uparrow)^{[\]}$, the contraction c_A is obtained by applying the Basic Perturbation Lemma. This technique will allow us to extend this result to the case of A being an A_∞ -algebra, obtaining a contraction between the reduced cobar construction of the tilde bar construction of A , $\bar{\Omega}\tilde{B}A$, and A .

Now we expose the main theorem of the paper. Afterwards, we will dualize the result for A_∞ -coalgebras.

Theorem 4.1. Let M be a DG-module endowed with an A_∞ -algebra structure. Then there exists a contraction, $c : \{A_M, M, f, g, \phi\}$, between a connected DG-algebra A_M and the DG-module M , such that the application of the tensor trick to c provides the original A_∞ -algebra structure on M .

Proof.

Let $(M, m_1, m_2, \dots)$ be an A_∞ -algebra and let $\{\bar{\Omega} T^c(s\bar{M}), M, \alpha, \rho, h\}$ be the contraction whose component morphisms are described ahead. In order to make clearer the formulas, we denote by c_i the i -th component an element of the cobar, $\langle c_1 | \dots | c_n \rangle$, with length $k(i) \geq 1$, that is, $c_i = [a_{i,1} | \dots | a_{i,k(i)}]$ and we will only specify the whole expression $[a_{i,1} | \dots | a_{i,k(i)}]$ in the case we want emphasize its composition. Particularly, when $k(i) = 1$, we write $[a_i]$.

- $\alpha : \bar{\Omega} T^c(s\bar{M}) \rightarrow M,$

$$\alpha \langle c_1 | \dots | c_n \rangle = 0 \text{ if } n > 1 \text{ or } k(1) > 1;$$

$$\alpha \langle [a] \rangle = a;$$

- $\rho : M \rightarrow \bar{\Omega} T^c(s\bar{M}),$

$$\rho(a) = \langle [a] \rangle$$

- $h : \bar{\Omega} T^c(s\bar{M}) \rightarrow \bar{\Omega} T^c(s\bar{M})$

$$h \langle c_1 | \dots | c_n \rangle = 0 \text{ if } n = 0 \text{ or } n = 1 \text{ or } k(1) > 1;$$

$$h \langle [a_1] | c_2 | \dots | c_n \rangle = (-1)^{|a_1|+1} \langle [a_1 | a_{2,1} | \dots | a_{2,k(2)}] | c_3 | \dots | c_n \rangle;$$

Now the perturbation δ will consist in including $(m_2, m_3, \dots)$ in the tensor differential of the cobar, so that $(\bar{\Omega} T^c(s\bar{M}), d_{\bar{\Omega}} + \delta)$, becomes $\bar{\Omega} \tilde{B}(M)$:

$$\delta = (\downarrow (- \sum_{k \geq 2} (\uparrow m_k \downarrow^{\otimes k})^{[1]}) \uparrow)^{[1]}.$$

The composition $h\delta$ is pointwise nilpotent, since δ decreases the simplicial degree of the components on the bar construction and h decreases the number of components on the cobar.

Then, we apply the Basic Perturbation Lemma, obtaining the contraction

$$c_M : \{\bar{\Omega} \tilde{B}(M), M, f, g, \phi\}.$$

Notice that for an element $\omega = \langle [a_1] | c_2 | \dots | c_n \rangle$

$$\delta h(\omega) = (-1)^{|a_1|+\zeta_{k(2)+1}} \langle [m_{k(2)+1}(a_1 \otimes a_{2,1} \otimes \dots \otimes a_{2,k(2)})] | c_3 | \dots | c_n \rangle$$

$$+ \text{ other summands with } k(1) > 1,$$

where $\zeta_{k(2)+1} = k(2)|a_1| + (k(2) - 1)|a_{2,1}| + \cdots + |a_{2,k(2)-1}| + [(k(2) + 1)/2]$.

Then, we can describe recursively $(\delta h)^i$, up to sign:

$$(\delta h)^i(\omega) = \langle [m_{k(i+1)+1}(\pi_{1,1}((\delta h)^{i-1}\omega) \otimes a_{i+1,1} \otimes \cdots \otimes a_{i+1,k(i+1)})] \\ |c_{i+2}| \cdots |c_n\rangle,$$

where $\pi_{1,1}\langle [a_{1,1}|\cdots|a_{1,k(1)}]|c_2|\cdots|c_n\rangle = a_{1,1}$.

Taking into account these notes, we can describe the morphisms of the contraction above:

- $f(\omega) = \pm m_{k(n)+1}(\pi_{1,1}((\delta h)^{n-2}\omega) \otimes a_{n,1} \otimes \cdots \otimes a_{n,k(n)})$.

In the particular case of the element $\langle [a_1]|c_2\rangle$,

$$f\langle [a_1]|c_2\rangle = (-1)^{1+|a_1|+\zeta_{k(2)+1}} m_{k(2)+1}(a_1 \otimes a_{2,1} \otimes \cdots \otimes a_{2,k(2)}). \quad (3)$$

- $g = \rho$, since $\delta\langle [a]\rangle = 0$.

- $\phi(\omega) = \sum_{i=1}^{n-1} (-1)^{i-1} h(\delta h)^{i-1} \langle [a_1]|c_2|\cdots|c_n\rangle$

$$= (-1)^{|a_1|+1} \langle [a_1|a_{2,1}|\cdots|a_{2,k(2)}]|c_3|\cdots|c_n\rangle$$

$$+ \sum_{i=2}^{n-1} \pm \langle [m_{k(i)+1}(\pi_{1,1}((\delta h)^{i-2}\omega) \otimes a_{i,1} \otimes \cdots \otimes a_{i,k(i)})|a_{i+1,1}|\cdots|a_{i+1,k(i+1)}]|c_{i+2}|\cdots|c_n\rangle$$

Particularly, for an element $\langle [a_1]|c_2\rangle$,

$$\phi\langle [a_1]|c_2\rangle = (-1)^{|a_1|+1} \langle [a_1|a_{2,1}|\cdots|a_{2,k(2)}]\rangle. \quad (4)$$

Besides, the perturbed differential on M is zero since $\delta\rho = 0$, what means that M still remains with the same differential structure.

We can check now that the structure of A_∞ -algebra generated on M via the tensor trick coincides with the original infinite structure on M .

Take the contraction $T(sc_M) : \{T(s\bar{\Omega}\widetilde{B}(M)), T(s\bar{M}), Tf, Tg, T(-\phi)\}$ and the simplicial differential, d_s , that depends on the juxtaposing product $\mu_{\bar{\Omega}}$, as a perturbation datum. The pointwise nilpotency of $T(-\phi)d_s$ is due to the fact that d_s does not modify the sum of the lengths of the components of the cobar, meanwhile $T(-\phi)$ decreases, at least by one, this amount. Then, applying the Basic Perturbation Lemma, the following contraction is obtained

$$(f_s, g_s, \phi_s) : \bar{B}\bar{\Omega}\widetilde{B}(M) \Rightarrow (T(s\bar{M}), \widetilde{d}).$$

In order to shorten the formulas ahead, we will denote the iterated composition of morphisms $\phi^{[\otimes i]} \mu^{(i)} \cdots \phi^{[\otimes j]} \mu^{(j)}$, with $i \leq j$, by $\Psi^{(i,j)}$ and $\phi^{[\otimes i]} \mu^{(i)}$, simply by $\Psi^{(i)}$. So, the family of

morphisms given in section 3 that provides an A_∞ -algebra structure for M can be expressed as follows:

$$\tilde{m}_n = (-1)^{n+1} f \mu^{(1)} \Psi^{(2,n-1)} g^{\otimes n}.$$

We will prove that each morphism $\tilde{m}_n$ is exactly the original m_n of the A_∞ -algebra structure on M . Take $a = a_1 \otimes \cdots \otimes a_n \in M^{\otimes n}$. Then,

$$\begin{aligned} \mu^{(n-1)} g^{\otimes n}(a) &= \sum_{i=0}^{n-2} (-1)^{i+1} \langle [a_1] \rangle \otimes \cdots \otimes \langle [a_i] \rangle \otimes \langle [a_{i+1}] \mid [a_{i+2}] \rangle \\ &\quad \otimes \langle [a_{i+3}] \rangle \otimes \cdots \otimes \langle [a_n] \rangle. \end{aligned}$$

The only non-null summand of $\phi^{[\otimes n-1]}$ that can be now applied on each summand on the right hand side is $1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes n-i-2}$, where ϕ is applied (following (4)) to the only element of cosimplicial dimension 2.

$$\begin{aligned} \phi^{[\otimes n-1]} \mu^{(n-1)} g^{\otimes n}(a) &= \sum_{i=0}^{n-2} (-1)^{i+1+\gamma_i} \langle [a_1] \rangle \otimes \cdots \otimes \langle [a_i] \rangle \otimes \langle [a_{i+1} \mid a_{i+2}] \rangle \\ &\quad \otimes \langle [a_{i+3}] \rangle \otimes \cdots \otimes \langle [a_n] \rangle, \end{aligned}$$

where $\gamma_i = |a_1| + \cdots + |a_i| + |a_{i+1}| + 1$.

Notice that the obtained summands have again, only factors of cosimplicial dimension 1 and this condition remains every time that $\Psi^{(k)}$, for any k , is applied: take an element from $(\bar{\Omega}\tilde{B}(M))^{\otimes j}$ in the form $\bar{\omega} = \langle c_1 \rangle \otimes \cdots \otimes \langle c_j \rangle$, then,

$$\mu^{(j-1)}(\bar{\omega}) = \sum_{i=0}^{j-2} (-1)^{i+1} \langle c_1 \rangle \otimes \cdots \otimes \langle c_i \rangle \otimes \langle c_{i+1} \mid c_{i+2} \rangle \otimes \langle c_{i+3} \rangle \otimes \cdots \otimes \langle c_j \rangle,$$

so $\phi^{[\otimes j-1]}$ (on each summand above) is reduced to $1^{\otimes i} \otimes \phi \otimes (gf)^{\otimes j-i-2}$ where ϕ is applied to the only factor of cosimplicial degree 2 and will be non-null only if $c_{i+1} = [a_{i+1}]$,

$$\Psi^{(j-1)}(\bar{\omega}) = \sum_{i=0}^{j-2} (-1)^{i+1+\gamma_i} \langle c_1 \rangle \otimes \cdots \otimes \langle c_i \rangle \otimes \langle [a_{i+1} \mid a_{i+2,1} \mid \cdots \mid a_{i+2,k(i+2)}] \rangle$$

$$\otimes g f \langle c_{i+3} \rangle \otimes \cdots \otimes g f \langle c_j \rangle, \quad (5)$$

with $\gamma_i = |c_1| + \cdots + |c_i| + i + |a_{i+1}| + 1$.

Besides, the elements $f \langle c_{i+3} \rangle, \dots, f \langle c_j \rangle$ will be non-null only if $k(i+3) = 1 = \cdots = k(j)$. In this case, the i -th summand of (5) will be

$$\langle c_1 \rangle \otimes \cdots \otimes \langle c_i \rangle \otimes \langle [a_{i+1} \mid a_{i+2,1} \mid \cdots \mid a_{i+2,k(i+2)}] \rangle \otimes \langle [a_{i+3}] \rangle \otimes \cdots \otimes \langle [a_j] \rangle,$$

which has only factors with 1 component.

This way, in order to obtain a non-null result for the i -th summand, the element to which $\Psi^{(j-1)}$ is applied must be

$$\langle c_1 \rangle \otimes \cdots \otimes \langle c_i \rangle \otimes \langle [a_{i+1}] \rangle \otimes \langle c_{i+2} \rangle \otimes \langle [a_{i+3}] \rangle \otimes \cdots \otimes \langle [a_j] \rangle. \quad (6)$$

As for the application of the whole sequence $\mu^{(1)}\Psi^{(2,n-1)}$, starting from a tensor product of n elements from the cobar construction, the morphism $\mu^{(i)}$ always decreases by one the number of factors from the cobar construction and $\phi^{[\otimes i]}$ does not touch this amount. This way, after applying $\mu^{(1)}\Psi^{(2,n-1)}$, only one factor is obtained to which f will be applied. Taking this into account together with the fact that the only elements that survive to $\Psi^{(j-1)}$ are those in the form (6), we conclude that the only summand of $\Psi^{(j-1)}$ that will pass through $f\mu^{(1)}\Psi^{(2,j-2)}$ is that of $i = j-2$ in the sum (5), whenever $c_{j-1} = [a_{j-1}]$. Then,

$$\Psi^{(j-1)}(\bar{\omega}) = (-1)^{j-1+\gamma_{j-2}} \langle c_1 \rangle \otimes \cdots \otimes \langle c_{j-2} \rangle \otimes \langle [a_{j-1}|a_{j,1}|\cdots|a_{j,k(j)}] \rangle + \cdots,$$

where dots represent the rest of summands that, from now on, we will omit.

Now, by induction on k , one can easily prove that

$$\Psi^{(n-k,n-1)} g^{\otimes n}(a) = (-1)^{\varrho(k)} \langle [a_1] \rangle \otimes \cdots \otimes \langle [a_{n-k-1}] \rangle \otimes \langle [a_{n-k}|\cdots|a_n] \rangle,$$

where $\varrho(k) = \sum_{i=0}^{k-1} (n-i) + \sum_{i=1}^k i |a_{n-i}| + k \sum_{i=k+1}^{n-1} |a_{n-i}|$.

So, for the case $k = n-2$,

$$\Psi^{(2,n-1)} g^{\otimes n}(a) = (-1)^{\varrho(n-2)} \langle [a_1] \rangle \otimes \langle [a_2|\cdots|a_n] \rangle,$$

where

$$\begin{aligned} \varrho(n-2) &= \sum_{i=0}^{n-3} (n-i) + \sum_{i=1}^{n-2} i |a_{n-i}| + (n-2)|a_1| \\ &\equiv [(n-1)/2] + \sum_{i=1}^{n-2} i |a_{n-i}| + (n-2)|a_1| \pmod{2}. \end{aligned}$$

Taking up again the calculation of $\tilde{m}_n$,

$$\begin{aligned} \tilde{m}_n(a_1 \otimes \cdots \otimes a_n) &= (-1)^{n+1+\varrho(n-2)} f \mu^{(1)} \langle [a_1] \rangle \otimes \langle [a_2|\cdots|a_n] \rangle \\ &= (-1)^{n+\varrho(n-2)} f \langle [a_1] | [a_2|\cdots|a_n] \rangle \end{aligned}$$

$$(\text{using 3}) = (-1)^{n+\varrho(n-2)+1+|a_1|+\zeta_n} m_n(a_1 \otimes \cdots \otimes a_n),$$

where $\zeta_n = \sum_{i=1}^{n-1} i |a_{n-i}| + [n/2]$ and hence

$$n + \varrho(n-2) + 1 + |a_1| + \zeta_n \equiv 0 \pmod{2}.$$

That is, the A_∞ -algebra structure obtained coincides with the original one defined on M . $\square$

This theorem provides a structural representation of an A_∞ -algebra as a contraction.

Corolary 4.2. *A DG-module M is endowed with an A_∞ -algebra structure if and only if there exists a contraction $c : \{A_M, M, f, g, \phi\}$, between a connected DG-algebra A_M and a DG-module M .*

Now, we show the dual result to the theorem given above, in the case of A_∞ -coalgebras, omitting its proof since it follows a similar scheme (though quite more tedious). From now on, all the DG-modules will be considered to be *simply connected*. This way, any A_∞ -coalgebra can be represented as a contraction from a simply connected DG-coalgebra.

Theorem 4.3. *Let M be a DG-module endowed with an A_∞ -coalgebra structure. Then there exists a contraction $c : \{C_M, M, f, g, \phi\}$, between a simply connected DG-coalgebra C_M and the DG-module M , such that the application of the tensor trick to c yields the initial A_∞ -coalgebra structure on M .*

In this case, the simply connection guarantees that the perturbation process is finite and a contraction $c_M : \{\tilde{B}\tilde{\Omega}(M), M, f, g, \phi\}$ can be constructed.

Corolary 4.4. *A DG-module M is endowed with an A_∞ -coalgebra structure if and only if there exists a contraction $c : \{C_M, M, f, g, \phi\}$, between a simply connected DG-coalgebra C_M and a DG-module M .*

We are aware that, at this point, the development of a categorical framework for A_∞ -(co)algebras in terms of chain contractions would be a natural direction of our future efforts.

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