Controllability of some systems of parabolic equations

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1. Introduction. Statement of the problem
1 Introduction. Statement of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial \Omega$ regular enough ($\Omega \in C^{0,1}$). Let $\omega, O \subseteq \Omega$ be two open subsets and fix $T > 0$. We consider the coupled parabolic system:

\begin{equation}
\begin{aligned}
\begin{cases}
y_t - \Delta y = q 1_O, & q_t - \Delta q = v 1_\omega \\
y = 0 & \text{on } \Sigma := \partial \Omega \times (0, T), \\
q = 0 & \text{on } \Sigma,
\end{cases}
\end{aligned}
\end{equation}

In (1), $1_\omega$ and $1_O$ are the characteristic functions of the sets $\omega$ and $O$, $(y(x, t), q(x, t))$ is the state, $(y_0, q_0) \in L^2(\Omega; \mathbb{R}^2)$ is the initial datum and $v \in L^2(Q)$ is the control function (which is localized in $\omega$ -distributed control-).
1 Introduction. Statement of the problem

Remark
In this talk we are interested in studying the controllability properties of system (1) (controllability to trajectories).

Definition
It will be said that system (1) is exactly controllable to trajectories at time $T$ if for any $(y_0, q_0) \in L^2(\Omega; \mathbb{R}^2)$ and any trajectory $(\tilde{y}, \tilde{q})$ of system (1) (a weak solution to (1) corresponding to $\tilde{v} \in L^2(Q)$ and $(\tilde{y}_0, \tilde{q}_0) \in L^2(\Omega; \mathbb{R}^2)$) there exists $v \in L^2(Q)$ s. t. the solution $(y, q)$ to (1) satisfies

$$(y(\cdot, T), q(\cdot, T)) \equiv (\tilde{y}(\cdot, T), \tilde{q}(\cdot, T)) \quad \text{in} \quad \Omega.$$ 

Remark
Using the linearity this concept is equivalent to the null controllability at time $T$ of system (1): For every $(y_0, q_0) \in L^2(\Omega; \mathbb{R}^2)$ there exists $v \in L^2(Q)$ s. t. the solution $(y, q)$ to (1) satisfies $(y(\cdot, T), q(\cdot, T)) \equiv 0$ in $\Omega$. 

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Controllability of some systems of parabolic equations
1 Introduction. Statement of the problem

IMPORTANT

We have a system of two coupled heat equations and we want to control the system (two states) only acting on the second equation.

A weaker concept:

Definition

It will be said that system (1) is approximately controllable at time $T$ if for any $(y_0, q_0) \in L^2(\Omega; \mathbb{R}^2)$, any $(y_d, q_d) \in L^2(\Omega; \mathbb{R}^2)$ and any $\varepsilon > 0$ there exists $v \in L^2(Q)$ s. t. the solution $(y, q)$ to (1) satisfies

$$\| (y(\cdot, T), q(\cdot, T)) - (y_d, q_d) \|_{L^2(\Omega)} \leq \varepsilon.$$
1 Introduction. Statement of the problem

\[
\begin{align*}
    y_t - \Delta y &= q_1 \mathcal{O}, & q_t - \Delta q &= v_1 \omega \text{ in } Q, \\
    y &= 0 \text{ on } \Sigma, & y(\cdot, 0) &= y_0 \text{ in } \Omega, \\
    q &= 0 \text{ on } \Sigma, & q(\cdot, 0) &= q_0 \text{ in } \Omega,
\end{align*}
\]

Existing results:

\[\omega \cap \mathcal{O} \neq \emptyset\] : Approximate and exact controllability to trajectories.

1 Introduction. Statement of the problem

\[
\begin{align*}
\begin{cases}
y_t - \Delta y &= q^1 \circ, \\
q_t - \Delta q &= v^1 \omega \\
y &= 0 \text{ on } \Sigma, \\
y(\cdot, 0) &= y_0 \text{ in } \Omega, \\
q &= 0 \text{ on } \Sigma, \\
q(\cdot, 0) &= q_0 \text{ in } \Omega,
\end{cases}
\end{align*}
\]

Existing results:

- $\omega \cap \mathcal{O} = \emptyset$: Approximate controllability.
- $\omega \cap \mathcal{O} = \emptyset$: Exact controllability to trajectories:
1 Introduction. Statement of the problem

\( \omega \cap \mathcal{O} = \emptyset \): Exact controllability to trajectories for a different parabolic system:

1 Introduction. Statement of the problem

Objective

We want to study the controllability properties of system (1) in the one-dimensional case $N = 1$.

We take $N = 1$, $\Omega = (0, \pi)$ and $\omega = (a, b)$, with $a, b \in (0, \pi)$, $a < b$. We consider the boundary controlled system

$$
\begin{align*}
\begin{cases}
    y_t - y_{xx} = q(x)A_0 y & \text{in } Q := (0, \pi) \times (0, T), \\
y(0, \cdot) = Bv, & \text{on } (0, T), \\
y(\pi, \cdot) = 0, & \text{on } (0, T), \\
y(\cdot, 0) = y_0, & \text{in } (0, \pi),
\end{cases}
\end{align*}
$$

where $y = (y_1(x, t), y_2(x, t))$ is the state, $v \in L^2(0, T)$ is the control (one boundary control) and $y_0 \in L^2(0, \pi; \mathbb{R}^2)$ is the initial datum. In (2), $q \in L^\infty(Q)$, $B = (b_1, b_2)^* \in \mathbb{R}^2$ and

$$
A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
$$

are given (in (1), $q = 1 \circ$ and $B = (0, 1)^*$).
2. Boundary controllability. Main result
2 Boundary controllability. Main result

\begin{equation}
\begin{aligned}
    y_t - y_{xx} &= q(x) A_0 y \quad \text{in } Q, \\
    y(0, \cdot) &= Bv, \quad y(\pi, \cdot) = 0 \quad \text{on } (0, T), \\
    y(\cdot, 0) &= y_0, \quad \text{in } (0, \pi),
\end{aligned}
\end{equation}

Let us consider the sequence of eigenvalues and normalized eigenfunctions of the operator $-\partial_{xx}$ on $(0, \pi)$ with homogenous Dirichlet boundary conditions:

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).$$

**Theorem**

Assume $\int_0^\pi q(x) \, dx \neq 0$. Then, system (2) is exactly controllable to trajectories at time $T$ if and only if $b_2 \neq 0$ and $I_k \neq 0 \quad \forall k \geq 1$, where

$$I_k := \int_0^\pi q(x) |\phi_k(x)|^2 \, dx.$$
2 Boundary controllability. Main result

Distributed control problem:

\[
\begin{cases}
 y_t - y_{xx} = A_0 y_1 + B v_1 & \text{in } Q, \\
 y(0, \cdot) = 0, & y(\pi, \cdot) = 0 & \text{on } (0, T), \\
 y(\cdot, 0) = y_0, & \text{in } (0, \pi),
\end{cases}
\]

Corollary

System (4) is exactly controllable to trajectories at time T if and only if \(b_2 \neq 0\).
2 Boundary controllability. Main result

Distributed control problem:

\[
\begin{cases}
y_t - y_{xx} = A_0 y_1 \mathcal{O} + B v 1_{\omega} & \text{in } Q, \\
y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\
y(\cdot, 0) = y_0, & \text{in } (0, \pi),
\end{cases}
\]

Corollary

System (4) is exactly controllable to trajectories at time T if and only if \( b_2 \neq 0 \).

Remark

1. The previous result is independent of the condition \( \mathcal{O} \cap \omega \neq \emptyset \).
2. We will focus on Theorem 3. Corollary 4 is a consequence of the boundary controllability result.
Boundary control problem: The proof of Theorem 3 is very technical and uses two main ingredients:

1. The moment method: 
   
2 Boundary controllability. Main result

Boundary control problem: The proof of Theorem 3 is very technical and uses two main ingredients:


2. Some results on existence and uniform bounds on biorthogonal families to complex exponentials:
2 Boundary controllability. Main result

Using the identity

\[(y(\cdot, T), \varphi_0)_{L^2(0, \pi; \mathbb{R}^2)} - (y_0, \varphi(\cdot, 0))_{L^2(0, \pi; \mathbb{R}^2)} = \int_0^\pi v(t) B^* \varphi_x(0, t) \, dt,\]

valid for every \(\varphi_0 \in L^2(0, \pi; \mathbb{R}^2)\), with \(\varphi\) the weak solution to the adjoint problem

\[
\begin{aligned}
-\varphi_t - \varphi_{xx} &= q(x) A_0^* \varphi \quad \text{in } Q, \\
\varphi(0, \cdot) &= 0, \quad \varphi(\pi, \cdot) = 0 \quad \text{on } (0, T), \\
\varphi(\cdot, T) &= \varphi_0, \quad \text{in } (0, \pi),
\end{aligned}
\]

the null controllability problem for system (2) amounts to:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\text{Find } v \in L^2(0, T) \text{ s.t. for every } \varphi_0 \in L^2(0, \pi; \mathbb{R}^2) \\
-(y_0, \varphi(\cdot, 0))_{L^2(0, \pi; \mathbb{R}^2)} = \int_0^\pi v(t) B^* \varphi_x(0, t) \, dt.
\end{array} \right.
\end{aligned}
\]
2 Boundary controllability. Main result

Idea

Use a basis of $L^2(0, \pi; \mathbb{R}^2)$ of eigenfunctions and generalized eigenfunctions of the operator

$$L := -\frac{d^2}{dx^2} - q(x)A_0 : L^2(0, \pi; \mathbb{R}^2) \rightarrow L^2(0, \pi; \mathbb{R}^2)$$

with domain $D(L) = H^2(0, \pi; \mathbb{R}^2) \cap H^1_0(0, \pi; \mathbb{R}^2)$. 
3. Spectral decomposition
Lemma

The spectrum of $L$ is given by $\sigma(L) = \{\lambda_k := k^2 : k \geq 1\}$. Moreover, $\lambda_k$ is simple if and only if $I_k \neq 0$ (see (3)). Finally, if $I_k = 0$, the eigenvalue $\lambda_k$ of $L$ is double.

Adjoint operator $L^*$

The same result can be obtained for the adjoint operator $L^* := -\frac{d^2}{dx^2} - q(x)A_0^*$. 
Proposition ($I_k \neq 0$ and operator $L$)

If

$$\Phi_{k,1} = \begin{pmatrix} \phi_k \\ 0 \end{pmatrix}, \quad \Phi_{k,2} = \begin{pmatrix} \psi_k \\ c_k \phi_k \end{pmatrix},$$

where $c_k = l_k^{-1}$ and $\psi_k$ is the unique solution of the problem:

$$\begin{cases}
-\psi_{xx} = \lambda_k \psi - (1 - c_k q(x)) \phi_k \text{ in } (0, \pi), \\
\psi(0) = 0, \quad \psi(\pi) = 0, \\
\int_0^\pi \psi(x) \phi_k(x) \, dx = 0,
\end{cases}$$

(7)

then,

$$(L - \lambda_k I_d) \Phi_{k,1} = 0, \quad (L - \lambda_k I_d) \Phi_{k,2} = -\Phi_{k,1}.$$
Proposition ($I_k = 0$ and operator $L$)

The eigenvalue $\lambda_k$ is double and the normalized eigenfunctions are given by

$$
\tilde{\Phi}_{k,1} = \Phi_{k,1} = \begin{pmatrix} \phi_k \\ 0 \end{pmatrix}, \quad \tilde{\Phi}_{k,2} = \frac{1}{C_k} \begin{pmatrix} \tilde{\psi}_k \\ \phi_k \end{pmatrix},
$$

where $\tilde{\psi}_k$ is the unique solution of the problem

$$
\begin{cases}
-\psi_{xx} = \lambda_k \psi + q(x) \phi_k & \text{in } (0, \pi), \\
\psi(0) = 0, \quad \psi(\pi) = 0, \\
\int_{0}^{\pi} \psi(x) \phi_k(x) \, dx = 0,
\end{cases}
$$

and $C_k = \sqrt{1 + \|\tilde{\psi}_k\|_{L^2(0,\pi)}^2}$. 

3 Spectral decomposition

Proposition ($I_k \neq 0$ and operator $L^*$)

If

$$\Phi^*_k,1 = \left( \begin{array}{c} \phi_k \\ \frac{1}{c_k} \psi_k \end{array} \right), \quad \Phi^*_k,2 = \left( \begin{array}{c} 0 \\ \frac{1}{c_k} \phi_k \end{array} \right),$$

($\psi$ is given by (7)) then,

$$(L^* - \lambda_k I_d) \Phi^*_k,1 = -\Phi^*_k,2, \quad (L^* - \lambda_k I_d) \Phi^*_k,2 = 0.$$ 

Proposition ($I_k = 0$ and operator $L^*$)

The eigenvalue $\lambda_k$ is double and the normalized eigenfunctions are given by

$$\tilde{\Phi}^*_k,1 = \frac{1}{C_k} \left( \begin{array}{c} \phi_k \\ \tilde{\psi}_k \end{array} \right), \quad \tilde{\Phi}^*_k,2 = \left( \begin{array}{c} 0 \\ \phi_k \end{array} \right),$$

where $\tilde{\psi}_k$ is the unique solution of (8) and $C_k = \sqrt{1 + \|\tilde{\psi}_k\|^2_{L^2(0,\pi)}}$. 
Assume $l_k \neq 0$ for any $k \geq 1$. Then, $\mathcal{B} = \{\Phi_{k,1}, \Phi_{k,2} : k \geq 1\}$ (resp. $\mathcal{B}^* = \{\Phi_{k,1}^*, \Phi_{k,2}^* : k \geq 1\}$) is a basis of $L^2(0, \pi; \mathbb{R}^2)$. 
4. Sketch of the proof of the main result
4 Sketch of the proof of the main result

\[
\begin{aligned}
y_t - y_{xx} &= q(x) A_0 y \quad \text{in } Q, \\
y(0, \cdot) &= Bv, \quad y(\pi, \cdot) = 0 \quad \text{on } (0, T), \\
y(\cdot, 0) &= y_0, \\
\end{aligned}
\]

(2)

Theorem

Assume \( \int_0^\pi q(x) \, dx \neq 0 \). Then, system (2) is exactly controllable to trajectories at time \( T \) if and only if \( b_2 \neq 0 \) and \( I_k \neq 0 \quad \forall k \geq 1 \), where

\[
I_k := \int_0^\pi q(x) |\phi_k(x)|^2 \, dx \quad (\phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad x \in (0, \pi)).
\]
4 Sketch of the proof of the main result

\[
\begin{cases}
y_t - y_{xx} = q(x) A_0 y & \text{in } Q, \\
y(0, \cdot) = Bv, & y(\pi, \cdot) = 0 & \text{on } (0, T), \\
y(\cdot, 0) = y_0, & \text{in } (0, \pi),
\end{cases}
\]

(2)

Theorem

Assume \( \int_0^\pi q(x) \, dx \neq 0 \). Then, system (2) is exactly controllable to trajectories at time \( T \) if and only if \( b_2 \neq 0 \) and \( I_k \neq 0 \) \( \forall k \geq 1 \), where

\[
l_k := \int_0^\pi q(x)|\phi_k(x)|^2 \, dx \quad (\phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad x \in (0, \pi)).
\]

Proof.

**Necessary condition**: If \( b_2 = 0 \) the second equation of system (2) is independent of the control \( v \) and, clearly, \( y_2 \) cannot be controlled. Assume then \( b_2 \neq 0 \).
4 Sketch of the proof of the main result

The exact controllability to trajectories at time $T$ of system (2) is equivalent to the **observability inequality** for the solutions to the **adjoint system** (5): $\exists C > 0$ s.t.

$$\| \varphi(\cdot, 0) \|_{L^2(0, \pi, \mathbb{R}^2)}^2 \leq C \int_0^T |B^* \varphi_x(0, t)|^2 \, dt, \quad \forall \varphi_0 \in L^2(0, \pi, \mathbb{R}^2).$$

By contradiction, let us suppose that $I_k = 0$ for some $k \geq 1$. Then $\lambda_k = k^2$ is an eigenvalue of $L^*$ of multiplicity 2 and $\tilde{\Phi}^*_k, 1$ and $\tilde{\Phi}^*_k, 2$ are the corresponding eigenfunctions. Let us consider $\varphi_0 = \alpha \tilde{\Phi}^*_k, 1 + \beta \tilde{\Phi}^*_k, 2$. The **adjoint system** (5) transforms into the o.d. system

$$-z' = \lambda_k z \text{ on } (0, T), \quad z(T) = (\alpha, \beta)^* \in \mathbb{R}^2,$$

which must satisfy the previous **observability inequality** and therefore the Kalman rank condition $\text{rank } [B | \lambda_k I_d B] = 2$. But this is absurd.
4 Sketch of the proof of the main result

**Sufficient condition:** \( l_k \neq 0 \) for all \( k \geq 1 \) and then (Proposition 9) \( \mathcal{B}^* = \{ \Phi^*_{k,1}, \Phi^*_{k,2} \}_{k \geq 1} \) is a basis of \( L^2(0, \pi; \mathbb{R}^2) \). Thus, in (6) we can take \( \varphi_0 \equiv \Phi^*_{k,1} \) and \( \varphi_0 \equiv \Phi^*_{k,2} \) obtaining an equivalent formulation to (6):

**I.** Taking \( \varphi_0 \equiv \Phi^*_{k,1} \) and \( \varphi_0 \equiv \Phi^*_{k,2} \), the corresponding solutions to the adjoint problem (5) are given

\[
\varphi_{k,1}(x, t) = e^{-\lambda_k(T-t)} \left[ \Phi^*_{k,1}(x) + (T-t)\Phi^*_{k,2}(x) \right],
\]

\[
\varphi_{k,2}(x, t) = e^{-\lambda_k(T-t)} \Phi^*_{k,2}(x).
\]

**II.** Replacing in (6), we deduce that system (2) is **null controllable** at time \( T \) if and only there exists \( v \in L^2(0, T) \) s.t., \( \forall k \geq 1 \),

\[
\begin{bmatrix}
 b_2 \\
 \sqrt{\frac{2}{\pi}} \frac{k}{c_k} v_{k,1}
\end{bmatrix} = -\frac{1}{c_k} e^{-\lambda_k T} (y_{0,2}, \phi_k)_{L^2(0,\pi)},
\]

\[
B^* \Phi^*_{k,1}(0) v_{k,1} + \begin{bmatrix}
 b_2 \\
 \sqrt{\frac{2}{\pi}} \frac{k}{c_k} v_{k,2}
\end{bmatrix} = -e^{-\lambda_k T} (y_0, \Phi^*_{k,1} + T \Phi^*_{k,2})_{L^2(0,\pi;\mathbb{R}^2)},
\]

where

\[
v_{k,1} = \int_0^T e^{-\lambda_k(T-t)} v(t) \, dt, \quad v_{k,2} = \int_0^T (T-t)e^{-\lambda_k(T-t)} v(t) \, dt.
\]
4 Sketch of the proof of the main result

The moment problem

Summarizing, system (2) is **null controllable** at time $T$ if and only if there exists $\tilde{v} \in L^2(0, T)$ ($v(t) = \tilde{v}(T - t), \forall t \in (0, T)$) s.t.

$$
\begin{align*}
\int_0^T e^{-\lambda_k t} \tilde{v}(t) \, dt &= M_{k,1}(y_0, q), \\
\int_0^T t e^{-\lambda_k t} \tilde{v}(t) \, dt &= M_{k,2}(y_0, q),
\end{align*}
$$

for any $k \geq 1$, where

$$
|M_{k,1}(y_0, q)| + |M_{k,2}(y_0, q)| \leq C(q) \left\| e^{-\lambda_k T} \right\| y_0 \left\|_{L^2},
$$

(we have used $\int_0^\pi q(x) \, dx \neq 0$).

We will solve the previous problem using a result from [1] (see also [2]).
4 Sketch of the proof of the main result

Lemma

There exists a sequence \( \{\varphi_{k,1}, \varphi_{k,2}\}_{k \geq 1} \) biorthogonal to \( \{e^{-\lambda_k t}, te^{-\lambda_k t}\}_{k \geq 1} \) in \( L^2(0, T) \):

\[
\begin{align*}
\int_0^T e^{-\lambda_k t} \varphi_{j,1}(t) \, dt &= \int_0^T te^{-\lambda_k t} \varphi_{j,2}(t) \, dt = \delta_{kj}, \\
\int_0^T te^{-\lambda_k t} \varphi_{j,1}(t) \, dt &= \int_0^T e^{-\lambda_k t} \varphi_{j,2}(t) \, dt = 0,
\end{align*}
\]

Moreover, for every \( \epsilon > 0 \) there is \( C_\epsilon(T) > 0 \) such that

\[
\left\| (\varphi_{k,1}, \varphi_{k,2}) \right\|_{L^2(0, T)} \leq C_\epsilon(T)e^{\epsilon\lambda_k}, \quad \forall k \geq 1.
\]
4 Sketch of the proof of the main result

**Lemma**

There exists a sequence \( \{ \varphi_{k,1}, \varphi_{k,2} \}_{k \geq 1} \) biorthogonal to \( \{ e^{-\lambda_k t}, te^{-\lambda_k t} \}_{k \geq 1} \) in \( L^2(0, T) \):

\[
\begin{align*}
\int_0^T e^{-\lambda_k t} \varphi_{j,1}(t) \, dt &= \int_0^T te^{-\lambda_k t} \varphi_{j,2}(t) \, dt = \delta_{kj}, \\
\int_0^T te^{-\lambda_k t} \varphi_{j,1}(t) \, dt &= \int_0^T e^{-\lambda_k t} \varphi_{j,2}(t) \, dt = 0,
\end{align*}
\]

Moreover, for every \( \varepsilon > 0 \) there is \( C_\varepsilon(T) > 0 \) such that

\[
\left\| (\varphi_{k,1}, \varphi_{k,2}) \right\|_{L^2(0, T)} \leq C_\varepsilon(T) e^{\varepsilon \lambda_k}, \quad \forall k \geq 1.
\]

**Solution**

\[
\tilde{v}(t) = \sum_{k \geq 1} \left[ M_{k,1}(y_0, q) \varphi_{k,1}(t) + M_{k,2}(y_0, q) \varphi_{k,2}(t) \right].
\]
Finally, \( \tilde{\nu} \in L^2(0, T) \) since the previous series converges in \( L^2(0, T) \). Indeed, taking \( \varepsilon = T/2 \), one has

\[
\| M_{k,1}(y_0, q)\varphi_{k,1} \|_{L^2} + \| M_{k,2}(y_0, q)\varphi_{k,2} \|_{L^2} \leq C(T)e^{-\lambda_k T/2} \| y_0 \|_{L^2}
\]

which guarantees the absolute convergence of the series in \( L^2(0, T) \). This ends the proof.
Thank you for your attention!!