Some examples of orthogonal matrix polynomials satisfying odd order differential equations

Antonio J. Durán and Manuel D. de la Iglesia

Departamento de Análisis Matemático. Universidad de Sevilla
Apdo (P. O. BOX) 1160. 41080 Sevilla. Spain.
duran@us.es, mdi29@us.es

Abstract

It is well known that if a finite order linear differential operator with polynomial coefficients has as eigenfunctions a sequence of orthogonal polynomials with respect to a positive measure (with support in the real line), then its order has to be even. This property no longer holds in the case of orthogonal matrix polynomials. The aim of this paper is to present examples of weight matrices such that the corresponding sequences of matrix orthogonal polynomials are eigenfunctions of certain linear differential operators of odd order. The weight matrices are of the form

$$W(t) = t^{\alpha}e^{-t}e^{At}t^{B}t^{B^*}e^{A^*t},$$

where A and B are certain (nilpotent and diagonal, respectively) $N \times N$ matrices. These weight matrices are the first examples illustrating this new phenomenon which are not reducible to scalar weights.

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1 Introduction

During the last few years many families of orthogonal matrix polynomials have been found that are eigenfunctions of some fixed second order linear differential operator with matrix coefficients which do not depend on the degree of the polynomial. When the corresponding eigenvalues are Hermitian then the differential operator for the orthonormal polynomials $(P_n)_n$ is also symmetric with respect to (the inner product defined from) its orthogonalizing weight

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matrix W, and conversely. More precisely, for an operator (whose coefficients are multiplied on the right)

$$\ell_2 = D^2 A_2(t) + D^1 A_1(t) + D^0 A_0, \tag{1}$$

and a family of orthonormal polynomials $(P_n)_n$ with respect to a weight matrix W, the following conditions are equivalent:

(1) $\ell_2(P_n(t)) = \Gamma_n P_n(t)$, with Γ_n Hermitian
(2) $\int \ell_2(P) dW Q^* = \int P dW \ell_2(Q)^*, \quad P, Q \in \mathbb{C}^{N\times N}[t]$.

An overview of those families of orthogonal matrix polynomials satisfying second order differential operators shows some important differences with respect to the scalar situation. First of all, it has been necessary to develop new techniques (rather different from the scalar ones) to find examples (see [5], [4], [10], [11] and [12]). Secondly, the complexity of the matrix world has led to an embarrassment of riches if we compare with the only scalar examples of Hermite, Laguerre and Jacobi. Also some new phenomena have appeared. For instance, some families of orthogonal polynomials $(P_n)_n$ have been found which are common eigenfunctions of several linearly independent second order differential operators (see [3], [9] or [8]). This shows that the algebras of differential operators having these families of orthogonal matrix polynomials as eigenfunctions are going to have a richer structure than the corresponding algebras in the scalar case, where for the classical scalar families those algebras reduce to the associated second order differential operator and any polynomial in this operator (see [17]).

The purpose of this paper is to show a new phenomenon. Indeed, in the scalar case it is well known that if a differential operator has a sequence of orthogonal polynomials as eigenfunctions then its order has to be even (see [14]). This situation is not true in the matrix case. We present here some examples of orthogonal matrix polynomials which are eigenfunctions of certain differential operators of odd order. Those examples are particular cases of orthogonal matrix polynomials with respect to a wider family of $N \times N$ weight matrices defined by

$$W_{\alpha,\nu_1,\cdots,\nu_{N-1}}(t) = t^\alpha e^{-t} e^{At} t^{\frac{1}{2}J} t^{\frac{1}{2}J^*} e^{A^* t}, \quad \alpha > -1, \quad t \in (0, +\infty), \tag{2}$$

where A is the $N \times N$ nilpotent matrix:

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \nu_i \in \mathbb{C} \setminus \{0\}, \quad i = 1, \cdots, N-1,$$

and J is the $N \times N$ diagonal matrix:

$$J = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The weight matrix corresponding to $N = 2$ yields the following third order symmetric differential operator:

$$\begin{aligned} \ell_{3,1} =& D^3 \begin{pmatrix} -|a|^2 t^2 & a t^2 (1 + |a|^2 t) \\ -\bar{a} t & |a|^2 t^2 \end{pmatrix} + \\ & D^2 \begin{pmatrix} -t(2 + |a|^2(\alpha + 5)) & at(2\alpha + 4 + t(1 + |a|^2(\alpha + 5))) \\ -\bar{a}(\alpha + 2) & t(2 + |a|^2(\alpha + 2)) \end{pmatrix} + \\ & D^1 \begin{pmatrix} t - 2(\alpha + 2)(1 + |a|^2) & \frac{|a|^2(\alpha+1)(\alpha+2)+t(1+2|a|^2(1+|a|^2(\alpha+2))}{\bar{a}} \\ -\frac{1}{a} & 2\alpha + 2 - t \end{pmatrix} + \\ & D^0 \begin{pmatrix} 1 + \alpha & -\frac{1}{\bar{a}}(1 + \alpha)(|a|^2\alpha - 1) \\ \frac{1}{a} & -(1 + \alpha) \end{pmatrix}. \end{aligned} \tag{3}$$

In fact, the weight matrix has two linearly independent symmetric third order differential operators and other two linearly independent operators of order two. The weight matrices corresponding to $N = 3$ and 4 yield to fifth and seventh order (respectively) symmetric differential operator, but no third order. This is related to the fact that A is nilpotent of order N. Our conjecture is that each $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ has a symmetric differential operator of order $2N - 1$, as well as other of even order (as we will see below, $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ always has at least one symmetric second order differential operator).

Some other examples of 2×2 situations have recently appeared in the literature having differential operators of odd order (order 1, to be more precise): see [2], [3] and [7]. Some of them correspond to positive definite weight matrices and happen to be symmetric. Some others do not even correspond to a positive definite weight matrix. However, the first ones have weights that reduce to scalar weights. We say that a weight matrix W reduces to scalar weights if there exists a nonsingular matrix T independent of t for which

$$W(t) = TD(t)T^*,$$

with $D(t)$ diagonal. Weight matrices reducible to scalar weights are actually a collection of N scalar weights. When they are considered by themselves and not in connection with differential equations these weights belong to the study of scalar orthogonality more than to the matrix one. The 2×2 positive definite weight matrices referred above (see [2], [3] and [7]) are of the form $W(t) = T \begin{pmatrix} \omega_1(t) & 0 \\ 0 & \omega_2(t) \end{pmatrix} T^*$, where T is a nonsingular matrix (independent of t). We have to point out that, however, this matrix T do not factorize their associated differential operators (that is, that operators are not of the form $T \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} T^*$; see [11] for a notion of scalar reducibility for the pair consisting of the weight and the differential operator). According to the previous definition, the examples shown in this paper do not reduce to scalar weight and hence are the first weight matrices with this property that have a symmetric differential operator of odd order.

The paper is organized as follows. In Section 2, we introduce the family $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ (see (2)) and study its symmetric second order differential operators. We find that

$$\ell_{2,1} = D^2 tI + D^1[(\alpha+1)I + J + t(A-I)] + D^0[(J+\alpha I)A - J]$$

is symmetric for $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$. When the moduli of the parameters ν_i, $i = 1, \cdots, N-2$, are defined from ν_{N-1} by

$$i(N-i)|\nu_{N-1}|^2 = (N-1)|\nu_i|^2 + (N-i-1)|\nu_i|^2|\nu_{N-1}|^2,$$

our weight matrix enjoys another symmetric second order differential operator $\ell_{2,2}$ with leading coefficient $A_2(t) = t(J - At)$. It turns out that the operators $\ell_{2,1}$ and $\ell_{2,2}$ commute and satisfy the polynomial equation

$$\prod_{i=1}^{N} \left((i-1)\ell_{2,1} - \ell_{2,2} + \left[\frac{(N-1)(N-i)}{|\nu_{N-1}|^2} + (i-1)(N-i) \right] I \right) = 0.$$

We note here that the symbol 0 is used to denote either zero scalar or zero matrix (all entries equal to zero scalar) while I is used to denote the identity matrix of dimension determined from the context.

We complete Section 2 showing some structural formulas (Rodrigues' formula and three–term recurrence relation) for a sequence $(\mathcal{P}_{n,\alpha,a})_n$ of orthogonal polynomials with respect to the weight matrix

$$W_{\alpha,a}(t) = t^{\alpha} e^{-t} \begin{pmatrix} t(1+|a|^2 t) & at \\ \bar{a}t & 1 \end{pmatrix}, \quad \alpha > -1, t > 0, \tag{4}$$

which is the case $N = 2$ of (2) (putting $\nu_1 = a$). This weight matrix appears for the first time in [1], where the authors found a Pearson equation for it and proved that the derivatives $(\mathcal{P}'_{n,\alpha,a})_n$ are again orthogonal with respect to a certain weight matrix.

In order to look for symmetric differential operators of higher order with respect to the weight matrix $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$, we show, in Section 3, how the symmetry of a differential operator ℓ_k of any order k with respect to a weight matrix W can be reduced to a set of $k+1$ differential equations (of order $0, 1, \cdots, k$, respectively) and certain boundary conditions for W and the coefficients of ℓ_k at the endpoints of the support of W. This can be seen as the matrix valued version of Littlejohn's conditions for the symmetry over polynomials of this kind of differential operators in the scalar case (see [15]).

Finally, in Section 4, we study the algebra of differential operators associated with (4) defined by

$$\mathcal{D}(W_{\alpha,a}) = \left\{ \ell = \sum_{i=0}^{k} D^i A_i(t) : \ell(\mathcal{P}_{n,\alpha,a}(t)) = \Gamma_n(\ell) \mathcal{P}_{n,\alpha,a}(t),\ n \geq 0 \right\}.$$

In particular, we find two linearly independent symmetric third order differential operators (one of them is given in (3)). It seems that two new linearly independent operators appear as one increases by one the order of the operators in question. We give strong computational evidences which support our conjecture that the algebra $\mathcal{D}(W_{\alpha,a})$ is generated by the set $\{I, \ell_{2,1}, \ell_{3,1}\}$ (except for some exceptional values involving the parameters α and a).

2 Second order differential operators for $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$

The aim of this section is to introduce the weight matrix $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ and study its second order differential operators.

Under the assumption that

$$A_2(t)W(t) \text{ and } (A_2(t)W(t))' - A_1(t)W(t),$$

vanish at each of the endpoints of the support of $W(t)$, to find symmetric second order differential operators for a weight matrix W is enough to solve the following equations:

$$\begin{align} A_2 W &= W A_2^*, \tag{5}\\ 2(A_2 W)' &= W A_1^* + A_1 W, \tag{6}\\ (A_2 W)'' - (A_1 W)' + A_0 W &= W A_0^*, \tag{7} \end{align}$$

(see Theorem 3.1 of [5] or also [11]).

Assuming that $A_2(t)$ is a scalar matrix, it has been proved in [5] that the differential equation (6) is equivalent to the fact that W can be factorized in the form $W(t) = \rho(t)T(t)T^*(t)$, where ρ is a scalar function and T is a matrix function satisfying certain first order differential equation. When $\rho(t) = t^\alpha e^{-t}$, $\alpha > -1$, i.e. the Laguerre classical scalar weight, and $A_2(t) = tI$, this first order differential equation for T takes the form

$$T'(t) = \left(A + \frac{B}{t}\right) T(t). \tag{8}$$

Let us consider a weight matrix of the form $W(t) = t^\alpha e^{-t} T(t)T(t)^*$, where

$$T(t) = e^{At} t^B = e^{At} e^{B \log t}, \tag{9}$$

with A and B any matrices. Using the formula

$$e^{At} B = \left(\sum_{n \geq 0} \frac{t^n}{n!} \operatorname{ad}_A^n B \right) e^{At},$$

we can write the derivative of (9) as

$$T'(t) = AT + \frac{1}{t} TB = \left(\frac{B}{t} + A + \operatorname{ad}_A B + \sum_{n \geq 2} \frac{t^{n-1}}{n!} \operatorname{ad}_A^n B \right) T.$$

We use the standard notation

$$\operatorname{ad}_X^0 Y = Y, \quad \operatorname{ad}_X^1 Y = [X, Y], \quad \operatorname{ad}_X^2 Y = [X, [X, Y]],$$

and, in general, $\operatorname{ad}_X^{n+1} Y = [X, \operatorname{ad}_X^n Y]$, where $[X, Y] = XY - YX$.

In order for T to satisfy a differential equation like (8), we need to choose matrices A and B for which $\mathrm{ad}_A^2 B = 0$. Once we have chosen the matrix

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \nu_i \in \mathbb{C} \setminus \{0\}, \quad i = 1, \cdots, N-1, \tag{10}$$

and B to be diagonal and singular, it follows from the condition $\mathrm{ad}_A^2 B = 0$ that necessarily $B = uJ$ where

$$J = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{11}$$

and u is any complex number. However, it turns out that for the existence of a symmetric second order differential operator with $A_2(t) = tI$, u has to be $1/2$, except for $N = 2$ where u can be any complex number.

This is the reason why we have chosen the weight matrix to be

$$W_{\alpha,\nu_1,\cdots,\nu_{N-1}}(t) = t^\alpha e^{-t} e^{At} t^{\frac{1}{2}J} t^{\frac{1}{2}J^*} e^{A^* t}, \quad \alpha > -1, \quad t \in (0, +\infty),$$

where A and J are defined by (10) and (11), respectively, as a candidate to have symmetric second order differential operators like (1) with $A_2(t) = tI$. Our choice allows to write $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}(t) = t^\alpha e^{-t} T(t) T(t)^*$, where $T(t) = e^{At} t^{\frac{1}{2}J}$ satisfies

$$T'(t) = \frac{1}{2}\left(\frac{J}{t} + A\right) T(t). \tag{12}$$

Actually $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ is not a bad candidate!

Theorem 1 *The second order differential operator*

$$\ell_{2,1} = D^2 tI + D^1[(\alpha+1)I + J + t(A - I)] + D^0[(J + \alpha I)A - J], \tag{13}$$

is symmetric with respect to $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$.

PROOF. We only need to verify equation (7). Using (6), that equation is equivalent to $(A_1 W - W A_1^*)' = 2(A_0 W - W A_0^*)$ (where, to simplify the notation, we remove the dependence on α and $\nu_1, \cdots, \nu_{N-1}$). Using the formulas

$$e^{At}J = Je^{At} - Ate^{At} \quad \text{and} \quad e^{A^*t}J = Je^{A^*t} + A^*te^{A^*t}$$

we obtain

$$\begin{aligned} JW - WJ =& t(AW - WA^*) \\ JAW - WA^*J =& (AW - WA^*) + (AJW - WJA^*). \end{aligned}$$

Also, as a consequence of the first equation above, we deduce

$$\begin{aligned} JW' - W'J &= AW - WA^* + t(AW' - W'A^*) \\ t(A^2W - W(A^*)^2) &= (JWA^* - AWJ) + (AJW - WJA^*). \end{aligned}$$

Hence, the equation $(A_1W - WA_1^*)' = 2(A_0W - WA_0^*)$ follows from these equalities taking into account the differential equation for $T(t)$ in (12). $\square$

For $N = 2$, the weight matrix $W(t) = t^\alpha e^{-t}e^{At}t^Bt^{B^*}e^{A^*t}$, where

$$A = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix},$$

has associated the symmetric second order differential operator

$$\ell_2 = D^2 \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + D^1 \begin{pmatrix} 2u + \alpha + 1 - t & 2tv(1-u) \\ 0 & \alpha + 1 - t \end{pmatrix} + D^0 \begin{pmatrix} -1 & v(1+\alpha) \\ 0 & 0 \end{pmatrix}.$$

As we wrote above for $N = 3$, in order to have a symmetric second order differential operator with $A_2(t) = tI$ it is necessary that $B = \frac{1}{2}J$.

We now search for another second order differential operator

$$\ell_{2,2} = D^2A_2(t) + D^2A_1(t) + D^0A_0$$

for the weight matrix $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$. We follow the lines of the method developed in [4]. This method consists of looking for *good factorizations* of $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ in the form $t^\alpha e^{-t}R(t)R^*(t)$. Under the assumption that $A_2W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ is Hermitian, by a good factorization for $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ we mean that the factor R satisfies the first order differential equation $R' = FR$, where

(1) The differential coefficient F is related to A_2 and A_1 by the equation

$$A_1 = A_2F + FA_2 + C, \tag{14}$$

where $C(t) = (t^\alpha e^{-t}A_2(t))'/t^\alpha e^{-t}$ and

(2) The matrix function

$$R^{-1}(FA_2F+F'A_2+FC-A_0)R \tag{15}$$

is Hermitian.

Then (1) guarantees that $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ satisfies equation (6) and (2) equation (7) (see [4], Section 2).

According to this approach, first of all, we have to look for the leading coefficient A_2 of the differential operator $\ell_{2,2}$. This coefficient has to satisfy $A_2W = WA_2^*$ (we are again removing the dependence on α and $\nu_1,\cdots,\nu_N$). The relation $AJ-JA=-A$ gives a very natural candidate for A_2. Indeed, $A_2W=WA_2^*$ is equivalent to $t^{-\frac{1}{2}J}e^{-At}A_2e^{At}t^{\frac{1}{2}J}$ being Hermitian. If we put

$$A_2(t)=t(J-At), \tag{16}$$

a straightforward computation gives

$$\begin{aligned} t^{-\frac{1}{2}J}e^{-At}(t(J-At))e^{At}t^{\frac{1}{2}J} =& t^{-\frac{1}{2}J}\left(\sum_{n\geq 0}\frac{(-1)^n t^{n+1}}{n!}\mathrm{ad}_A^n J - At^2\right)t^{\frac{1}{2}J} \\ =& t\cdot t^{-\frac{1}{2}J}Jt^{\frac{1}{2}J}=tJ, \end{aligned}$$

implying that (5) is satisfied since tJ is Hermitian.

Once we have a candidate for A_2, we need to choose a certain good factorization of the weight matrix $W(t)=t^\alpha e^{-t}R(t)R^*(t)$ (in the sense explained above). We proceed by taking a unitary matrix function $U(t)$, and writing

$$R(t)=e^{At}t^{\frac{1}{2}J}U(t),$$

so that $W(t)=t^\alpha e^{-t}R(t)R(t)^*$. The matrix function $U(t)$ will be introduced later.

The definition of $R(t)$ implies that the coefficient F of the first order differential equation $R'(t)=F(t)R(t)$ for R is

$$F(t)=\frac{1}{2}\left(\frac{1}{t}J+A\right)+e^{At}t^{\frac{1}{2}J}X(t)t^{-\frac{1}{2}J}e^{-At}, \tag{17}$$

where we have written $X(t)=U'(t)U(t)^{-1}$. Since $U(t)$ is unitary, it turns out that the matrix function $X(t)$ is skew-Hermitian. Taking this into account we choose our matrix $X(t)$ to have the following structure:

$$X(t) = \begin{pmatrix} 0 & x_{1,2}(t) & 0 & \cdots & 0 & 0 \\ -\bar{x}_{1,2}(t) & 0 & x_{2,3}(t) & \cdots & 0 & 0 \\ 0 & -\bar{x}_{2,3}(t) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x_{N-1,N}(t) \\ 0 & 0 & 0 & \cdots & -\bar{x}_{N-1,N}(t) & 0 \end{pmatrix}, \tag{18}$$

where the complex functions $x_{i,i+1}(t)$, $i = 1, \cdots, N-1$, will be chosen so that equation (14) yields that A_1 is a polynomial of degree 1. From the definitions of F, A_2 and A_1 (see (17), (16) and (14)) it follows that

$$\begin{aligned} A_1 =&((1+\alpha)I + J)J - t(J + (\alpha+2)A) + t^2(A - A^2) + \\ & \qquad te^{At}t^{\frac{1}{2}J}(JX(t) + X(t)J)t^{-\frac{1}{2}J}e^{-At}. \end{aligned} \tag{19}$$

We now compute the right hand side of the expression above. From the definitions of $X(t)$ and J (see (18) and (11)) it follows that

$$X(t)J + JX(t) = \begin{pmatrix} 0 & y_{1,2}(t) & 0 & \cdots & 0 & 0 \\ -\bar{y}_{1,2}(t) & 0 & y_{2,3}(t) & \cdots & 0 & 0 \\ 0 & -\bar{y}_{2,3}(t) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & y_{N-1,N}(t) \\ 0 & 0 & 0 & \cdots & -\bar{y}_{N-1,N}(t) & 0 \end{pmatrix},$$

where

$$y_{i,i+1}(t) = (2(N-i)-1)x_{i,i+1}(t), \quad i = 1, \cdots, N-1.$$

Then, by direct calculation we have

$$M(t) = t^{\frac{1}{2}J}(X(t)J + JX(t))t^{-\frac{1}{2}J} =$$

$$= \begin{pmatrix} 0 & t^{\frac{1}{2}}y_{1,2}(t) & \cdots & 0 & 0 \\ -t^{-\frac{1}{2}}\bar{y}_{1,2}(t) & 0 & \cdots & 0 & 0 \\ 0 & -t^{-\frac{1}{2}}\bar{y}_{2,3}(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & t^{\frac{1}{2}}y_{N-1,N}(t) \\ 0 & 0 & \cdots & -t^{-\frac{1}{2}}\bar{y}_{N-1,N}(t) & 0 \end{pmatrix}.$$

Taking into account the definition of A (see (10)), this implies that

$$\mathrm{ad}_A M = \begin{pmatrix} z_{1,1}(t) & 0 & z_{1,3}(t) & \cdots & 0 & 0 \\ 0 & z_{2,2}(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & z_{3,3}(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & z_{N-2,N}(t) \\ 0 & 0 & 0 & \cdots & z_{N-1,N-1}(t) & 0 \\ 0 & 0 & 0 & \cdots & 0 & z_{N,N}(t) \end{pmatrix},$$

where (put $\nu_0 = 0$ and $\nu_N = 0$)

$$z_{i,i}(t) = (\nu_{i-1}\bar{y}_{i-1,i}(t) - \nu_i\bar{y}_{i,i+1}(t))t^{-\frac{1}{2}}, \quad i = 1, \cdots, N,$$

and

$$z_{i,i+2}(t) = (\nu_i y_{i+1,i+2}(t) - \nu_{i+1}y_{i,i+1}(t))t^{\frac{1}{2}}, \quad i = 1, \cdots, N-2.$$

In order for A_1 to be a matrix polynomial of degree 1 we need to take

$$y_{i,i+1}(t) = y_{i,i+1}t^{-\frac{1}{2}},$$

where now, abusing notation, $y_{i,i+1} \in \mathbb{C}$. We can then write $M(t) = \frac{1}{t}Y - Y^*$, where

$$Y = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -\bar{y}_{1,2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\bar{y}_{2,3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -\bar{y}_{N-1,N} & 0 \end{pmatrix}. \tag{20}$$

Hence, substituting all these expressions in (19) we have

$$\begin{aligned} A_1 = {} & ((1+\alpha)I + J)J + Y - t(J + (\alpha+2)A + Y^* - \mathrm{ad}_A Y) \\ & + t^2(A - A^2 + \frac{1}{2}\mathrm{ad}_A^2 Y - \mathrm{ad}_A Y^*) + \sum_{n\geq 3} \frac{t^n}{n!}(\mathrm{ad}_A^n Y - \mathrm{ad}_A^{n-1} Y^*). \end{aligned}$$

Since the structure of the matrices A and Y implies that $\mathrm{ad}_A^2 Y$ has null entries out of the diagonal $(i, i+1)$ and that $\mathrm{ad}_A Y^*$ has null entries out of the diagonal $(i, i+2)$, A_1 is a matrix polynomial of degree 1 if and only if

$$\begin{aligned} \tfrac{1}{2}\mathrm{ad}_A^2 Y + A &= 0 \\ \mathrm{ad}_A Y^* + A^2 &= 0. \end{aligned} \tag{21}$$

From the first equation in (21) we obtain $y_{i,i+1} = -\dfrac{i(N-i)}{\bar{\nu}_i}$ which implies

$$x_{i,i+1} = -\frac{i(N-i)}{(2N-2i-1)\bar{\nu}_i}, \quad i = 1, \cdots, N-1. \tag{22}$$

We are again abusing notation taking $x_{i,i+1}(t) = x_{i,i+1}t^{-\frac{1}{2}}$, $x_{i,i+1} \in \mathbb{C}$. From the second equation in (21) we get $y_{i,i+1} = \dfrac{\nu_i c}{\nu_1} + (i-1)\nu_i$ which implies

$$x_{i,i+1} = \frac{\nu_i c}{(2N-2i-1)\nu_1} + \frac{(i-1)\nu_i}{2N-2i-1}, \quad i = 1, \cdots, N-1, \tag{23}$$

where c is any complex number. Then, equating (22) and (23) we have the following set of equations:

$$c|\nu_i|^2 + (i-1)\nu_1|\nu_i|^2 + i(N-i)\nu_1 = 0, \quad i = 1, \cdots, N-1.$$

After removing the parameter c, we can write these equations in the following two equivalent and more convenient ways

$$i(N-i)|\nu_{i+1}|^2 = (i+1)(N-i-1)|\nu_i|^2 + |\nu_i|^2|\nu_{i+1}|^2, \tag{24}$$

$$i(N-i)|\nu_{N-1}|^2 = (N-1)|\nu_i|^2 + (N-i-1)|\nu_i|^2|\nu_{N-1}|^2, \tag{25}$$

for $i = 1, \cdots, N-2$, $N > 2$. With this choice of relations between the parameters $\nu_1, \cdots, \nu_{N-1}$, the function A_1 defined in (14) is a matrix polynomial of degree 1, namely:

$$A_1(t) = ((1+\alpha)I + J)J + Y - t(J + (\alpha+2)A + Y^* - \mathrm{ad}_A Y).$$

We now show that under the assumption (24) (or, equivalently (25)), we can produce a matrix A_0 such that W also satisfies the differential equation (7).

According to (15), we are going to prove that there exists a matrix A_0 such that the function

$$R^{-1}(t)(F(t)A_2(t)F(t) + F'(t)A_2(t) + F(t)(t^\alpha e^{-t}A_2)' t^{-\alpha}e^t - A_0)R(t)$$

is Hermitian. Since $R(t) = e^{At}t^{\frac{1}{2}J}U(t)$ and $U(t)$ is unitary, it is equivalent to prove that

$$\chi(t) = t^{-\frac{1}{2}J}e^{-At}(F(t)A_2(t)F(t) + F'(t)A_2(t) + F(t)(t^\alpha e^{-t}A_2)' t^{-\alpha}e^t - A_0)e^{At}t^{\frac{1}{2}J} \tag{26}$$

is always Hermitian.

We now look for a convenient expression for (26). We use the formulas

$$\begin{aligned} t^{-\frac{1}{2}J}e^{-At}(A_2)e^{At}t^{\frac{1}{2}J} &= t \cdot t^{-\frac{1}{2}J}Jt^{\frac{1}{2}J} = tJ,\\ t^{-\frac{1}{2}J}e^{-At}(A_2')e^{At}t^{\frac{1}{2}J} &= J - t^{\frac{1}{2}}A,\\ t^{-\frac{1}{2}J}At^{\frac{1}{2}J} &= t^{-\frac{1}{2}}A,\\ X(t) = t^{-\frac{1}{2}}X \quad \text{and} \quad X'(t) &= -\frac{1}{2t}X(t), \end{aligned} \tag{27}$$

where, again, abusing notation, X is the skew-Hermitian matrix independent of t with null entries except for the diagonals $(i, i+1)$ and $(i+1, i)$, whose entries are given in (22). From the definition of F and A_2 (see (17) and (16)), and after a straightforward computation, we obtain

$$\begin{aligned} t^{-\frac{1}{2}J}e^{-At}(FA_2F)e^{At}t^{\frac{1}{2}J} =& \frac{1}{4t}J^3 + \frac{1}{2}t^{-\frac{1}{2}}(J^2A + AJ^2) + AJA + tX(t)JX(t)\\ &+ t^{\frac{1}{2}}(AJX(t) + X(t)JA) + \frac{1}{2}(J^2X(t) + X(t)J^2),\\ t^{-\frac{1}{2}J}e^{-At}(F'A_2)e^{At}t^{\frac{1}{2}J} =& -\frac{1}{2t}J^2 + t^{\frac{1}{2}}(AX(t)J - X(t)AJ)\\ &- \frac{1}{2}X(t)J - \frac{1}{2}X(t)J^2 + \frac{1}{2}JX(t)J,\\ t^{-\frac{1}{2}J}e^{-At}(F(t^\alpha e^{-t}A_2)' t^{-\alpha}e^t)e^{At}t^{\frac{1}{2}J} =& \frac{\alpha+1}{2t}J^2 + t^{-\frac{1}{2}}((\alpha+1)AJ - \frac{1}{2}JA) - A^2 - \frac{1}{2}J^2\\ &- t^{\frac{1}{2}}(AJ + X(t)A) + (\alpha+1)X(t)J - tX(t)J. \end{aligned}$$

Using again (27), we obtain an expression for the matrix function $\chi(t)$ (see (26)):

$$\chi(t) = \frac{1}{t}\left[\frac{1}{4}J^2(J+2\alpha I)\right] + \frac{1}{2}t^{-\frac{1}{2}}\left[J^2A + AJ^2 + J^2X + (2\alpha+1)AJ + (2\alpha+1)XJ - \right.$$
$$\left. JA + JXJ\right] + AJA + XJX + AJX + XJA + AXJ - XAJ - XA -$$
$$A^2 - \frac{1}{2}J^2 - t^{\frac{1}{2}}(AJ + XJ) - t^{-\frac{1}{2}J}e^{-At}(A_0)e^{At}t^{\frac{1}{2}J}.$$

We define the matrix A_0 as follows

$$A_0 = \begin{pmatrix} A_{0,1,1} & A_{0,1,2} & 0 & \cdots & 0 & 0 \\ 0 & A_{0,2,2} & A_{0,2,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{0,N-1,N-1} & A_{0,N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & A_{0,N,N} \end{pmatrix} = V_1 + V_2,$$

where $V_{1,i,i} = A_{0,i,i}$, $i = 1, \cdots, N$, and $V_{1,i,j} = 0$ otherwise, and $V_2 = A_0 - V_1$. Then, we obtain:

$$t^{-\frac{1}{2}J}A_0t^{\frac{1}{2}J} = V_1 + t^{-\frac{1}{2}}V_2 \quad \text{and}$$
$$-t \cdot t^{-\frac{1}{2}J}(\operatorname{ad}_A A_0)t^{\frac{1}{2}J} = -t^{\frac{1}{2}}V_3 - V_4,$$

where

$$V_{3,i,i+1} = \begin{cases} \nu_i(A_{0,i+1,i+1} - A_{0,i,i}), & i = 1, \cdots, N-1, \\ 0, & \text{otherwise;} \end{cases}$$
$$V_{4,i,i+2} = \begin{cases} \nu_i A_{0,i+1,i+2} - \nu_{i+1}A_{0,i,i+1}, & i = 1, \cdots, N-2, \\ 0, & \text{otherwise.} \end{cases}$$

Taking this into account, as well as the fact that $X(JA - AJ) = XA$ and that $\frac{1}{4}J^2(J+2\alpha I)$, XJX and $-\frac{1}{2}J^2$ are already Hermitian, in order to prove that (26) is Hermitian it is enough to impose that the matrices

$$\frac{1}{2}[J^2A + AJ^2 + J^2X + (2\alpha+1)AJ + (2\alpha+1)XJ - JA + JXJ] - V_2, \tag{28}$$

$$-(AJ + XJ) - V_3 \quad \text{and} \tag{29}$$

$$(AJA + AJX + AXJ - A^2) - V_4 \tag{30}$$

are Hermitian and that

$$\operatorname{ad}_A^2 A_0 = 0. \tag{31}$$

Condition (28) allows us to define the upper diagonal $(i, i+1)$ of the matrix A_0 by

$$A_{0,i,i+1} = (\alpha + N - i)[\nu_i(N - i - 1) + x_{i,i+1}(2N - 2i - 1)], \quad i = 1, \cdots, N-1,$$

which, using (22) and (25), implies

$$A_{0,i,i+1} = -\frac{(N-1)(\alpha+N-i)\nu_i}{|\nu_{N-1}|^2}, \quad i = 1, \cdots, N-1. \tag{32}$$

Condition (29) gives us a recursive expression for the the elements $A_{0,i,i}$:

$$A_{0,i,i} = A_{0,i+1,i+1} - \left[N-i-1+(2N-2i-1)\frac{x_{i,i+1}}{\nu_i}\right], \quad i = N-1, N-2, \cdots, 1.$$

Putting $A_{0,N,N} = 0$ and using (22) and (25) we obtain

$$A_{0,i,i} = \frac{(N-1)(N-i)}{|\nu_{N-1}|^2}, \quad i = 1, \cdots, N. \tag{33}$$

Condition (30) is then equivalent to

$$(N-i-2)\nu_i\nu_{i+1} + (2N-2i-3)\nu_i x_{i+1,i+2} + \nu_i A_{0,i+1,i+2} - \nu_{i+1}A_{0,i,i+1} = 0,$$

which can be easily deduced using (22), (32) and (24). Finally, condition (31) is equivalent to

$$\begin{aligned}
(\mathrm{ad}_A^2 A_0)_{i,i+2} &= \nu_i\nu_{i+1}(A_{0,i,i} - 2A_{0,i+1,i+1} + A_{0,i+2,i+2}) = 0,\\
(\mathrm{ad}_A^2 A_0)_{i,i+3} &= \nu_i\nu_{i+1}A_{0,i+2,i+3} - 2\nu_i\nu_{i+2}A_{0,i+1,i+2} + \nu_{i+1}\nu_{i+2}A_{0,i,i+1} = 0,
\end{aligned}$$

for $i = 1, \cdots, N-2$, which can be deduced from (32) and (33).

By looking directly at (32) and (33) it is easy to conclude that

$$A_0 = \frac{N-1}{|\nu_{N-1}|^2}[J - (\alpha I + J)A].$$

We have thus proved the following theorem:

Theorem 2 *Let W be the weight matrix defined by (2) where the moduli of the entries $|\nu_i|, i = 1, \cdots, N-2$, of the matrix A are defined from ν_{N-1} using the equations (25). Consider the matrices A and J defined by (10) and (11) respectively, and Y, $X(t)$ and $F(t)$ defined by*

$$Y = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ \frac{N-1}{\nu_1} & 0 & 0 & \cdots & 0 & 0\\ 0 & \frac{2(N-2)}{\nu_2} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & \frac{N-1}{\nu_{N-1}} & 0 \end{pmatrix}, \tag{34}$$

$$X(t) = t^{-\frac{1}{2}} \begin{pmatrix} 0 & x_{1,2} & 0 & \cdots & 0 & 0 \\ -\bar{x}_{1,2} & 0 & x_{2,3} & \cdots & 0 & 0 \\ 0 & -\bar{x}_{2,3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x_{N-1,N} \\ 0 & 0 & 0 & \cdots & -\bar{x}_{N-1,N} & 0 \end{pmatrix},$$

$$F(t) = \frac{1}{2}\left(\frac{1}{t}J + A\right) + e^{At}t^{\frac{1}{2}J}X(t)t^{-\frac{1}{2}J}e^{-At},$$

where $x_{i,i+1} = -\dfrac{i(N-i)}{(2N-2i-1)\bar{\nu}_i}, \quad i = 1, \cdots, N-1$. *Finally define the coefficients of the differential operator* A_2, A_1 *and* A_0 *by:*

$$\begin{aligned} A_2 &= t(J - At), \\ A_1 &= ((1+\alpha)I + J)J + Y - t(J + (\alpha+2)A + Y^* - ad_A Y), \\ A_0 &= \frac{N-1}{|\nu_{N-1}|^2}[J - (\alpha I + J)A]. \end{aligned}$$

Then the weight matrix W *satisfies the set of equations*

$$\begin{aligned} A_2 W &= W A_2^*, \\ 2(A_2 W)' &= W A_1^* + A_1 W, \\ (A_2 W)'' - (A_1 W)' + A_0 W &= W A_0^*, \end{aligned}$$

so that the second order differential operator

$$\ell_{2,2} = D^2 A_2 + D^1 A_1 + D^0 A_0 \tag{35}$$

is symmetric with respect to W. W *can be factorized as* $W(t) = t^\alpha e^{-t} R(t) R^*(t)$, $\alpha > -1$, *where* $R(t) = e^{At} t^{\frac{1}{2}J} e^{2tX(t)}$ *satisfies* $R'(t) = F(t)R(t)$.

□

We now prove that the symmetric second order operators $\ell_{2,1}$ and $\ell_{2,2}$ (see (13) and (35)) for W commute and satisfy a relation given by a polynomial of degree N in two variables.

Theorem 3 *The second order differential operators* $\ell_{2,1}$ *and* $\ell_{2,2}$ *given in (13) and (35), respectively, commute, i.e.* $[\ell_{2,1}, \ell_{2,2}] = 0$. *Moreover, they satisfy the following relation:*

$$\prod_{i=1}^{N} \left((i-1)\ell_{2,1} - \ell_{2,2} + \left[\frac{(N-1)(N-i)}{|\nu_{N-1}|^2} + (i-1)(N-i) \right] I \right) = 0. \tag{36}$$

PROOF. Taking the monic orthogonal polynomials with respect to W as eigenfunctions, there exists an isomorphism between differential operators and their corresponding eigenvalues. Since the operators $\ell_{2,1}$ and $\ell_{2,2}$ have a common system of eigenfunctions, they commute if and only if their corresponding eigenvalues commute. These eigenvalues are given by:

$$\begin{aligned}
\Gamma_{n,1} &= n(A-I)+(J+\alpha I)A-J \\
\Gamma_{n,2} &= -n^2A-n(J+(\alpha+1)A+Y^*-\mathrm{ad}_AY)+\frac{N-1}{|\nu_{N-1}|^2}(J-(\alpha I+J)A).
\end{aligned}$$

Since $\Gamma_{n,1}$ and $\Gamma_{n,2}$ have non null entries only in the diagonals (i,i) and $(i,i{+}1)$, they commute if and only if

$$\begin{aligned}
(\Gamma_{n,1})_{i,i+1}[(\Gamma_{n,2})_{i+1,i+1}-(\Gamma_{n,2})_{i,i}]+(\Gamma_{n,2})_{i,i+1}[(\Gamma_{n,1})_{i,i}-(\Gamma_{n,1})_{i+1,i+1}] &= 0, \\
(\Gamma_{n,1})_{i,i+1}(\Gamma_{n,2})_{i+1,i+2}-(\Gamma_{n,2})_{i,i+1}(\Gamma_{n,1})_{i+1,i+2} &= 0.
\end{aligned}$$

These formulas can be easily checked using the definitions of A, J and Y (see (10), (11) and (34)) and formulas (24) and (25). Hence, $[\ell_{2,1},\ell_{2,2}]=0$.

Now we prove (36). We can use the already mentioned correspondence between differential operators and their eigenvalues. Let us write, for $i=1,\cdots,N$,

$$\Delta_i=(i-1)\Gamma_{n,1}-\Gamma_{n,2}+\left(\frac{(N-1)(N-i)}{|\nu_{N-1}|^2}+(i-1)(N-i)\right)I.$$

Then, it is enough to prove that $\Delta_1\Delta_2\cdots\Delta_N=0$. It is easy to check that the i-th element of the main diagonal of Δ_i is 0. Taking into account that $\Gamma_{n,1}$ and $\Gamma_{n,2}$ have non null entries only in the diagonals (i,i) and $(i,i+1)$, we deduce that, for $i=1,\cdots,N-1$, the i-th row of Δ_i has all its entries equal to zero except the entry $i+1$, and the row N of Δ_N vanishes. Then, it is straightforward to conclude that (36) holds. $\square$

We complete this section illustrating some structural formulas for a sequence of orthogonal polynomials $(\mathcal{P}_{n,\alpha,a})_n$ with respect to the weight matrix

$$W_{\alpha,a}(t)=t^\alpha e^{-t}\begin{pmatrix} t(1+|a|^2t) & at \\ \bar{a}t & 1\end{pmatrix}, \quad a\in\mathbb{C}\setminus\{0\}, \quad \alpha>-1, \quad t\in(0,+\infty), \tag{37}$$

which is the special case of the weight matrix $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$ for $N=2$.

Indeed, the sequence $(\mathcal{P}_{n,\alpha,a})_n$ can be defined by means of a Rodrigues' formula. Let us write

$$R_a(t)=\begin{pmatrix} t(1+|a|^2t) & at \\ \bar{a}t & 1\end{pmatrix}.$$

Then, the matrix polynomials defined by

$$\begin{aligned}\mathcal{P}_{0,\alpha,a} &= \begin{pmatrix} 1 & -a(1+\alpha) \\ 0 & 1 \end{pmatrix}, \\ \mathcal{P}_{n,\alpha,a}(t) &= \Phi_{n,\alpha,a}\Big[t^{\alpha+n}e^{-t}(R_a(t)+X_{n,a})\Big]^{(n)} R_a^{-1}(t)t^{-\alpha}e^{t}, \quad n \geq 1, \end{aligned} \tag{38}$$

where

$$\Phi_{n,\alpha,a} = \begin{pmatrix} 1 & -a(1+\alpha) \\ 0 & 1/\gamma_{n,a} \end{pmatrix}, \quad X_{n,a} = \begin{pmatrix} 0 & -an \\ 0 & 0 \end{pmatrix}, \quad \text{and}$$

$$\gamma_{n,a} = 1 + n|a|^2, \tag{39}$$

are orthogonal with respect to the weight matrix $W_{\alpha,a}$ defined in (37). The leading coefficient $\Lambda_{n,\alpha,a}$ is a nonsingular matrix given by

$$\Lambda_{n,\alpha,a} = (-1)^n \begin{pmatrix} 1 & -a(1+n+\alpha) \\ 0 & 1 \end{pmatrix}.$$

This can be proved as in [6] or [8].

The normalization by the matrices $\Phi_{n,\alpha,a}$ allows us to get simpler expressions for some other formulas related to polynomials $(\mathcal{P}_{n,\alpha,a})_n$. For instance, one can prove that they satisfy the following three–term recurrence relation:

$$t\mathcal{P}_{n,\alpha,a}(t) = D_n\mathcal{P}_{n+1,\alpha,a}(t) + E_n\mathcal{P}_{n,\alpha,a}(t) + F_n\mathcal{P}_{n-1,\alpha,a}(t),$$

where the matrices E_n, D_n, $n \geq 0$, and F_n, $n \geq 1$, are given explicitly by the expressions:

$$\begin{aligned} D_n &= -\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \\ E_n &= \begin{pmatrix} \dfrac{\gamma_{n+1,a}(2n+3+\alpha)-1}{\gamma_{n+1,a}} & a(1+n+\alpha) \\ \dfrac{\bar{a}}{\gamma_{n+1,a}\gamma_{n,a}} & \dfrac{\gamma_{n,a}(2n+\alpha)+1}{\gamma_{n,a}} \end{pmatrix}, \\ F_n &= -\frac{1}{\gamma_{n,a}}\begin{pmatrix} n\gamma_{n+1,a}(1+n+\alpha) & 0 \\ \dfrac{n\bar{a}}{\gamma_{n,a}} & n\gamma_{n-1,a}(n+\alpha) \end{pmatrix}. \end{aligned}$$

The L^2-norm of $(\mathcal{P}_{n,\alpha,a})_n$ is given by

$$\|\mathcal{P}_{n,\alpha,a}\|^2_{L^2(W)} = n!\begin{pmatrix} \Gamma(\alpha+n+2)\gamma_{n+1,a} & 0 \\ 0 & \Gamma(\alpha+n+1)/\gamma_{n,a} \end{pmatrix}.$$

We will use the family $(\mathcal{P}_{n,\alpha,a})_n$ in Section 4 to study the algebra of differential operators associated with the weight matrix $W_{\alpha,a}$. Before, we need some conditions of symmetry for higher order differential operators.

3 Conditions of symmetry for higher order differential operators

The purpose of this section is to generalize the symmetry conditions (5), (6) and (7) of a second order differential operator to symmetry conditions for a higher order differential operator with respect to a weight matrix W (for the scalar case see [16]).

Let ℓ_k be a differential operator of order k

$$\ell_k = \sum_{i=0}^{k} D^i A_i(t), \tag{40}$$

where D^i is the i–th derivative with respect to t and $A_i(t), i = 0, \cdots, k$, are matrix polynomials of degree no bigger than i. Let us write $A_i(t)$ as:

$$A_i(t) = \sum_{j=0}^{i} t^j A_j^i, \quad A_j^i \in \mathbb{C}^{N\times N}.$$

Firstly we relate the symmetry of ℓ_k with the moments $\mu_n = \int_\Omega t^n W(t)dt$, $n \geq 0$, of a weight matrix W, where Ω is the support of W. In this section, $[n]_i$ will denote the bounded factorial defined by

$$\begin{aligned} [n]_i =& n(n-1)\cdots(n-i+1), \quad n \geq i > 0, \\ [n]_0 =& 1, \quad [n]_i = 0, \quad i > n \geq 0. \end{aligned} \tag{41}$$

Proposition 4 *The differential operator ℓ_k (see (40)) is symmetric with respect to the weight matrix W if and only if the moments $(\mu_n)_n$ of W satisfy the following $k+1$ sets of equations:*

$$\sum_{i=0}^{k-l} \binom{k-i}{l} [n-l]_{k-l-i} B_n^{k-i} = (-1)^l (B_n^l)^*, \quad l = 0, \cdots, k, \quad n \geq l \tag{42}$$

where

$$B_n^l = \sum_{i=0}^{l} A_{l-i}^l \mu_{n-i}, \quad l = 0, \cdots, k, \quad n \geq l. \tag{43}$$

PROOF. The symmetry of ℓ_k implies

$$\int_\Omega \ell_k(t^n I)\, W(t)\, (t^m I)^*\, dt = \int_\Omega (t^n I)\, W(t)\, \ell_k(t^m I)^*\, dt, \quad n, m \geq 0. \tag{44}$$

Then, from (44) we have the following moment equations:

$$\sum_{i=0}^{k}[n]_{k-i}B_{n+m}^{k-i}=\sum_{i=0}^{k}[m]_{k-i}(B_{n+m}^{k-i})^{*}. \tag{45}$$

In order to obtain (42) we use bounded complete induction over l. When $l=0$ we put $m=0$ in (45) and we get the first equation.

Assume the first $j-1$ equations are true and let us show that the j–th also holds. Put $n-j+1$ and $m=j-1$ in (45) to get

$$\sum_{i=0}^{k}[n-j+1]_{k-i}B_{n}^{k-i}=\sum_{i=0}^{j-1}[j-1]_{i}(B_{n}^{i})^{*}. \tag{46}$$

Substitute the expressions for $(B_n^i)^*,\ i=0,\cdots,j-2$, defined in (42), into (46) and group again all the $B_n^i,\ i=0,\cdots,k$. To conclude it is enough to take into account that for $m=0,\cdots,k$, we have that

$$[n-j+1]_m=\sum_{h=0}^{m}(-1)^h\begin{pmatrix}m\\h\end{pmatrix}[j-1]_h[n-h]_{m-h}.$$

Note that, from the definition of bounded factorial given in (41), the left hand side of the formula above vanishes for $n=j+p-1$, where $p=0,\cdots,m-1$. If we see that the right hand side of this expression is also zero for all such values of n it will imply the equality since both expressions are polynomials in n.

Write the right hand side of the formula above as

$$n\cdots(n-m+1)\sum_{h=0}^{m}(-1)^h\begin{pmatrix}m\\h\end{pmatrix}\frac{(j-1)\cdots(j-h)}{n\cdots(n-h+1)}.$$

Substitute n by $j+p-1$, for $p=0,\cdots,m-1$ in the expression above. Then we have to verify that

$$(j+p-1)\cdots(j+p-m)\sum_{h=0}^{m}(-1)^h\begin{pmatrix}m\\h\end{pmatrix}\frac{(j-1)\cdots(j-h)}{(j+p-1)\cdots(j+p-h)}=0.$$

But this is always true considering the fact that $\sum_{h=0}^{m}(-1)^h\begin{pmatrix}m\\h\end{pmatrix}h^p=0$ for $p=0,\cdots,m-1$ and that the formula above is always a polynomial in h of degree at most $m-1$.

The converse is similar. □

Using the previous proposition, we can guarantee the symmetry of ℓ_k with respect to W from a set of differential equations which relates W with the coefficients of the differential operator ℓ_k.

Theorem 5 *Assume the boundary conditions that*

$$\sum_{i=0}^{p-1}(-1)^{k-i+p-1}\binom{k-i}{l}\Big(A_{k-i}\cdot W\Big)^{(p-1-i)}, \quad p=1,\cdots,k, \quad l=0,\cdots,k-p \tag{47}$$

should have vanishing limits at each of the endpoints of the support of W and that the following $k+1$ equations are satisfied:

$$\sum_{i=0}^{k-l}(-1)^{k-i-l}\binom{k-i}{l}\Big(A_{k-i}\cdot W\Big)^{(k-i-l)} = (-1)^l W\cdot A_l^*, \quad l=0,\cdots,k. \tag{48}$$

Then the differential operator ℓ_k (see (40)) is symmetric with respect to W.

PROOF. Using the moment equations (43) we obtain

$$[n-l]_{k-l-i}B_n^{k-i} = [n-l]_{k-l-i}\left(\sum_{j=0}^{k-i}A_{k-i-j}^{k-i}\mu_{n-j}\right) =$$
$$= \int_\Omega [n-l]_{k-l-i}t^{n-k+i}A_{k-i}(t)\cdot W(t)dt.$$

Integrating by parts $k-i-l$ times ($i\le k-l-1$) the previous integral is equal to

$$\sum_{j=1}^{k-i-l}(-1)^{j-1}[n-l]_{k-l-i-j}t^{n-k+i+j}\Big(A_{k-i}\cdot W\Big)^{(j-1)}\Bigg]_{\partial\Omega} +$$
$$(-1)^{k-i-l}\int_\Omega t^{n-l}\Big(A_{k-i}\cdot W\Big)^{(k-i-l)}dt.$$

Replacing the value of $[n-l]_{k-l-i}B_n^{k-i}$ in (42), we have to prove that

$$\sum_{i=0}^{k-l-1}\binom{k-i}{l}\left(\sum_{j=1}^{k-i-l}(-1)^{j-1}[n-l]_{k-l-i-j}t^{n-k+i+j}\Big(A_{k-i}\cdot W\Big)^{(j-1)}\Bigg]_{\partial\Omega}\right)$$
$$+\int_\Omega t^{n-l}\left(\sum_{i=0}^{k-l}(-1)^{k-i-l}\binom{k-i}{l}\Big(A_{k-i}\cdot W\Big)^{(k-i-l)} - (-1)^l W\cdot A_l^*\right)dt = 0.$$

But

(1) $\displaystyle\sum_{i=0}^{k-l-1}\binom{k-i}{l}\left(\sum_{j=1}^{k-i-l}(-1)^{j-1}[n-l]_{k-l-i-j}t^{n-k+i+j}\Big(A_{k-i}\cdot W\Big)^{(j-1)}\Bigg]_{\partial\Omega}\right)=0,$

for all $n-l\geq 0$. This follows expanding the sum, grouping in terms of the powers of t, taking into account the boundary conditions (47) and the fact that W must have finite moments.

(2) $\displaystyle\int_\Omega t^{n-l}\left(\sum_{i=0}^{k-l}(-1)^{k-i-l}\binom{k-i}{l}\Big(A_{k-i}\cdot W\Big)^{(k-i-l)}-(-1)^l W\cdot A_l^*\right)dt=0,$

for all $n-l\geq 0$, because of the symmetry equations (48).

$\square$

For $k=2$, Theorem 5 gives the conditions (5), (6) and (7) for the symmetry of a second order differential operator.

For $k=3$ those conditions are

$$\begin{aligned} A_3W+WA_3^*&=0,\\ -3(A_3W)'+A_2W&=WA_2^*,\\ -3(A_3W)''+2(A_2W)'&=WA_1^*+A_1W,\\ -(A_3W)'''+(A_2W)''-(A_1W)'+A_0W&=WA_0^*, \end{aligned}\tag{49}$$

and

$A_3W,\quad 3(A_3W)'-2(A_2W),\quad -(A_3W)'+(A_2W),\quad (A_3W)''-(A_2W)'+A_1W$

should vanish at the endpoints of the support of W.

4 The algebra of differential operators associated with $W_{\alpha,a}$

The main goal of this section is to study the structure of the algebra $\mathcal{D}(W_{\alpha,a})$ of differential operators having the family $(\mathcal{P}_{n,\alpha,a})_n$ of orthogonal polynomials with respect to $W_{\alpha,a}$ (given in (37)) as common eigenfunctions:

$$\mathcal{D}(W_{\alpha,a})=\left\{\ell=\sum_{i=0}^{k}D^iA_i(t):\ell(\mathcal{P}_{n,\alpha,a}(t))=\Gamma_n(\ell)\mathcal{P}_{n,\alpha,a}(t),\ n\geq 0\right\},$$

where $\Gamma_n(\ell)$ does not depend on t.

The product in $\mathcal{D}(W_{\alpha,a})$ is the composition of operators in the usual way, which it is not commutative. If $\ell_r,\ell_s\in\mathcal{D}(W_{\alpha,a})$ are of order r and s respectively, the operator $\ell_r \ell_s$ is of order less than or equal to $r+s$, but not necessarily $r+s$, since the leading coefficients of both operators can be singular. $\mathcal{D}(W_{\alpha,a})$ is both a complex vector space and an algebra under composition which is associative and distributive with respect to the field we are working.

The operators in $\mathcal{D}(W_{\alpha,a})$ need not to be symmetric with respect to $W_{\alpha,a}$. However, for every non symmetric operator $\ell \in \mathcal{D}(W_{\alpha,a})$, we can associate an adjoint operator $\ell^* \in \mathcal{D}(W_{\alpha,a})$ so that $\ell + \ell^*$ is symmetric with respect to $W_{\alpha,a}$. The previous assertion is a general fact for the algebra of operators associated with any weight matrix W. As a consequence, each operator $\ell \in \mathcal{D}(W_{\alpha,a})$ can be written uniquely in the form $\ell = \ell_1 + \imath \ell_2$ for certain symmetric operators ℓ_1, ℓ_2 (see [13] for details).

Using the fact that the leading coefficient of $\mathcal{P}_n(t)$ (to simplify the notation we remove the dependence on α and a) is a non singular matrix, we can see that each coefficient $A_i(t)$, $i = 1, \cdots, k$, of a differential operator ℓ in $\mathcal{D}(W)$ has to be a matrix polynomial of degree less than or equal to i. We observe that the map between differential operators and the corresponding eigenvalues given by

$$\Gamma_n : \mathcal{D}(W) \longrightarrow \mathbb{C}^{2\times 2}, \quad n = 0, 1, 2, \dots$$

is a faithful representation. The property $\Gamma_n(\ell_1 \ell_2) = \Gamma_n(\ell_1)\Gamma_n(\ell_2)$ with $\ell_1, \ell_2 \in \mathcal{D}(W)$ is easy to show as well as the fact that if $\Gamma_n(\ell) = 0$, $n \geq 0$, then $\ell = 0$.

It is important to note that the algebra $\mathcal{D}(W)$ is independent from the choice of the family of orthogonal matrix polynomials. The eigenvalues are changed by an n dependent conjugation. The family we use in this section is the sequence of orthogonal polynomials $(\mathcal{P}_n)_n$ defined by the Rodrigues' formula (38).

In the following table, obtained by direct computations, we exhibit the number of new linearly independent differential operators (modulo operators of lower order) that appear as one increases the order of the operators in question:

order	0	1	2	3	4	5	6	7	8
dimension	1	0	2	2	2	2	2	2	2

There are no first order differential operators. The system $\ell_{2,1}$ and $\ell_{2,2}$ of two linearly independent (symmetric) second order differential operators found in

Section 2, namely

$$\ell_{2,1} = D^2 tI + D^1 \begin{pmatrix} \alpha+2-t & at \\ 0 & \alpha+1-t \end{pmatrix} + D^0 \begin{pmatrix} -1 & (1+\alpha)a \\ 0 & 0 \end{pmatrix},$$

$$\ell_{2,2} = D^2 \begin{pmatrix} t & -at^2 \\ 0 & 0 \end{pmatrix} + D^1 \begin{pmatrix} \alpha+2 & -\frac{1}{\bar a}(1+|a|^2(\alpha+2))t \\ \frac{1}{a} & -t \end{pmatrix} + D^0 \begin{pmatrix} \frac{1}{|a|^2} & -\frac{1+\alpha}{\bar a} \\ 0 & 0 \end{pmatrix},$$

is a basis for the operators in $\mathcal{D}(W_{\alpha,a})$ of order 2. Their respective eigenvalues are given by

$$\Gamma_{n,2,1} = \begin{pmatrix} -n-1 & 0 \\ 0 & -n \end{pmatrix}, \quad \Gamma_{n,2,2} = \begin{pmatrix} \frac{1}{|a|^2} & 0 \\ 0 & -n \end{pmatrix}.$$

There are two linearly independent (symmetric) third order differential operators (modulo operators of lower order). As we explained in the introduction, this is a phenomenon which does not happen in the scalar case. In fact, this is the first example of weight matrix which does not reduce to scalar weights having symmetric differential operators of odd order.

A basis for the operators of order 3 in $\mathcal{D}(W_{\alpha,a})$ (modulo operators of lower order) is given by the operator $\ell_{3,1}$ defined in (3) and by $\ell_{3,2}$ defined as

$$\begin{aligned}
\ell_{3,2} =& D^3 \begin{pmatrix} |a|^2t^2 & at^2(-1+|a|^2t) \\ \bar a t & -|a|^2t^2 \end{pmatrix} + \\
& D^2 \begin{pmatrix} |a|^2t(\alpha+5) & -at(-2\alpha-4+t(3+|a|^2(\alpha+5))) \\ \bar a(\alpha+2) & -|a|^2t(\alpha+2) \end{pmatrix} + \\
& D^1 \begin{pmatrix} 2|a|^2(\alpha+2)+t & a(\alpha+1)(\alpha+2) - t\left(\frac{1}{\bar a} + 2a(2+|a|^2)(\alpha+2)\right) \\ -\frac{1}{a} & -t \end{pmatrix} + \\
& D^0 \begin{pmatrix} 1+\alpha & -\frac{1}{\bar a}(1+\alpha)(1+|a|^2(\alpha+2)) \\ \frac{1}{a} & -(1+\alpha) \end{pmatrix}.
\end{aligned}$$

Their respective eigenvalues are given by (recall the definition of $\gamma_{n,a}$ given in (39)):

$$\Gamma_{n,3,1} = \frac{1}{|a|^2} \begin{pmatrix} 0 & a(1+\alpha+n)\gamma_{n,a}\gamma_{n+1,a} \\ \bar a & 0 \end{pmatrix}, \quad \Gamma_{n,3,2} = \frac{1}{|a|^2} \begin{pmatrix} 0 & -a(1+\alpha+n)\gamma_{n,a}\gamma_{n+1,a} \\ \bar a & 0 \end{pmatrix}.$$

The operator $\ell_{3,1}$ is symmetric and hence satisfies the third order symmetry equations given by (49) with their corresponding boundary conditions. The operator $\ell_{3,2}$ is not symmetric but skew–symmetric so that $\imath\ell_{3,2}$ is symmetric.

The leading coefficients $A_{3,i}(t)$, $i = 1, 2$, of both operators can be written as

$$A_{3,1}(t) = t\left(At - \sum_{n\geq 0} \frac{t^n}{n!}\mathrm{ad}_A^n A^*\right), \quad A_{3,2}(t) = t\left(At + \sum_{n\geq 0} \frac{t^n}{n!}\mathrm{ad}_A^n A^*\right). \tag{50}$$

From those expressions it is easy to verify (using $t^{-\frac{1}{2}J}At^{\frac{1}{2}J} = t^{-\frac{1}{2}}A$, $t^{-\frac{1}{2}J}A^*t^{\frac{1}{2}J} = t^{\frac{1}{2}}A^*$ and $\sum_{n\geq 0}\frac{t^n}{n!}\mathrm{ad}_A^n A^* e^{At} = e^{At}A^*$) that

$$t^{-\frac{1}{2}J}e^{-At}(A_{3,1})e^{At}t^{\frac{1}{2}J} = t^{-\frac{1}{2}J}(At^2 - tA^*)t^{\frac{1}{2}J} = t^{\frac{3}{2}}(A - A^*),$$

implying that it verifies the first of the third order symmetry equations (49) since $t^{\frac{3}{2}}(A - A^*)$ is skew- Hermitian. An analogous argument is valid for $A_{3,2}$.

In order to propose a candidate for a system of operators which generate the full algebra $\mathcal{D}(W)$ and have easier expressions for relations between generators, let us introduce a different basis for the differential operators of order 2:

$$\begin{aligned} L_1 &= \ell_{2,1} - \frac{1}{|a|^2}I, \\ L_2 &= 2\ell_{2,2} - \ell_{2,1} - \frac{1}{|a|^2}I, \end{aligned}$$

while for the operators of order 3 we write $L_3 = \ell_{3,1}$ and $L_4 = \ell_{3,2}$. The corresponding system of eigenvalues associated with each operator is given by (recall again the definition of $\gamma_{n,a}$ given in (39))

$$\begin{aligned} \Gamma_{n,1} &= -\frac{1}{|a|^2}\begin{pmatrix} \gamma_{n+1,a} & 0 \\ 0 & \gamma_{n,a} \end{pmatrix}, \\ \Gamma_{n,2} &= \frac{1}{|a|^2}\begin{pmatrix} \gamma_{n+1,a} & 0 \\ 0 & -\gamma_{n,a} \end{pmatrix}, \\ \Gamma_{n,3} &= \frac{1}{|a|^2}\begin{pmatrix} 0 & a(1+\alpha+n)\gamma_{n,a}\gamma_{n+1,a} \\ \bar{a} & 0 \end{pmatrix}, \\ \Gamma_{n,4} &= \frac{1}{|a|^2}\begin{pmatrix} 0 & -a(1+\alpha+n)\gamma_{n,a}\gamma_{n+1,a} \\ \bar{a} & 0 \end{pmatrix}. \end{aligned}$$

The set $\{I, L_1, L_2, L_3, L_4\}$ is linearly independent and from the previous expressions for their eigenvalues it follows easily that they satisfy a number of relations, namely:

(1) Four quadratic relations:

$$\begin{aligned} L_1^2 &= L_2^2, \\ L_3^2 &= -L_4^2, \\ L_1L_2 &= L_2L_1, \\ L_3L_4 &= -L_4L_3. \end{aligned} \tag{51}$$

(2) Four permutational relations:

$$\begin{aligned} L_1L_3 - L_2L_4 &= 0, \\ L_2L_3 - L_1L_4 &= 0, \\ L_3L_2 + L_4L_1 &= 0, \\ L_3L_1 + L_4L_2 &= 0. \end{aligned} \tag{52}$$

(3) Four more second degree relations:

$$\begin{aligned} L_3 &= L_1L_4 - L_4L_1, \\ L_4 &= L_1L_3 - L_3L_1, \\ L_3 &= L_2L_3 + L_3L_2, \\ L_4 &= L_2L_4 + L_4L_2. \end{aligned} \tag{53}$$

(4) Finally, we can present two interesting third degree relations:

$$\begin{aligned} L_1L_3^2 &= L_3^2L_1, \\ L_2L_3^2 &= L_3^2L_2. \end{aligned} \tag{54}$$

The second equation in (53) allows us to remove the operator L_4 as a generator of the algebra $\mathcal{D}(W)$.

Computational evidences allows us to conclude that a possible basis for the operators of even order $2k$ in $\mathcal{D}(W)$ (modulo operators of lower order) is given by L_1^k and $L_1^{k-1}L_2$, while for operators of odd order one can take $L_1^kL_3 - L_3L_1^k$ and $L_2^kL_3 + L_3L_2^k$ for $4k-1, k \geq 1$, and $L_1^kL_3 + L_3L_1^k$ and $L_2^kL_3 - L_3L_2^k$ for $4k+1, k \geq 1$.

We finally present one more relation which allows us to conjecture that the full algebra of differential operators is generated by $\{I, L_1, L_3\}$. Indeed,

$$\begin{aligned} \Big[|a|^2(2+\alpha)-1\Big]\Big[|a|^2(\alpha-1)-1\Big] L_2 = {}& 2|a|^2\Big[|a|^2(2\alpha+1)-2\Big] L_1 \\ &+ \Big[|a|^4(\alpha^2+\alpha-5) - |a|^2(2\alpha+1)+1\Big] L_1^2 \\ &- 2|a|^2\Big[|a|^2(2\alpha+1)-2\Big] L_1^3 + 3|a|^4 L_1^4 \\ &- \frac{1}{2}\Big[|a|^2(2\alpha+1)-2\Big] L_3^2 + \frac{15}{2}|a|^2 L_3^2 L_1 - \frac{9}{2}|a|^2 L_3L_1L_3. \end{aligned}$$

Note that for the exceptional values of $\alpha = 1 + \frac{1}{|a|^2}$ or $\alpha = -2 + \frac{1}{|a|^2}$, the left side of the previous formula vanishes. For these cases, computational evidences allow us to conjecture that the algebra is generated by $\{I, L_1, L_2, L_3\}$.

If one is tempted to study the algebra of differential operators for higher matrix dimensions, the following table illustrates the number of linearly independent differential operators that appear as one increases the matrix dimension and order of the differential operators:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$N=2$	1	0	2	2	2	2	2	2	2
$N=3$	1	0	2	0	3	4	5	6	16
$N=4$	1	0	2	0	3	0	4	6	9

It is interesting that for $N = 3, 4$, there are no third order differential operators, but there are fifth and seventh order, respectively. The reason could be that the coefficients (50) for these cases are matrix polynomials of degree at least 4 (due to the fact that A is a nilpotent matrix of order N).

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