Second order differential operators having several families of orthogonal matrix polynomials as eigenfunctions*

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Abstract

The aim of this paper is to bring into the picture a new phenomenon in the theory of orthogonal matrix polynomials satisfying second order differential equations. The last few years have witnessed some examples of a (fixed) family of orthogonal matrix polynomials whose elements are common eigenfunctions of several linearly independent second order differential operators. We show that the dual situation is also possible: there are examples of one parametric families of monic matrix polynomials, each family orthogonal with respect to a different weight matrix, whose elements are eigenfunctions of a common second order differential operator.

These examples are constructed by adding a discrete mass to a weight matrix at a certain point. In this article it is described how to choose a point t_0, a discrete mass $M(t_0)$ and the weight matrix W so that the new weight matrix $W + \delta_{t_0} M(t_0)$ inherits some of the symmetric second order differential operators associated with W. It is well known that this situation is not possible for the classical scalar families of Hermite, Laguerre and Jacobi.

For some of these examples we characterize the convex cone of weight matrices for which the differential operator is symmetric.

1 Introduction

The theory of matrix valued orthogonal polynomials starts with two papers by M. G. Kreĭn in 1949, see [K1, K2]. A sequence of orthonormal matrix polynomials $(P_n)_n$ can be characterized as solutions of the difference equation

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n = 0, 1, \dots,$$

where A_n and B_n are $N \times N$ nonsingular and Hermitian matrices, respectively, and initial conditions $P_{-1} = 0$ and P_0 nonsingular. Each family $(P_n)_n$ goes along with a weight matrix W and satisfies $\int P_n dW P_m^* = \delta_{n,m} I$.

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More than 50 years later the first examples of orthogonal matrix polynomials $(P_n)_n$ satisfying second order differential equations of the form

$$P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0 = \Gamma_n P_n(t), \quad n = 0, 1, \ldots, \tag{1.1}$$

were produced. Here F_2, F_1 and F_0 are matrix polynomials (which do not depend on n) of degrees less than or equal to 2, 1 and 0, respectively (see [DG1, G, GPT3]). Two main methods have been developed in the last five years to produce such examples: solving an appropriate set of differential equations (see [D2, DG1, DG4, DdI]) or coming from the study of matrix valued spherical functions (see [GPT1, GPT4, PT]). In the case of matrix orthogonality, these families of orthogonal matrix polynomials are likely to play the role of the classical families of Hermite, Laguerre and Jacobi in the case of scalar orthogonality. The complexity of the matrix world however proved to be much richer compared to the scalar case (see, for instance, the papers cited above).

We point out here that, if the eigenvalues Γ_n are Hermitian, then the second order differential equation (1.1) for the orthonormal polynomials $(P_n)_n$ is equivalent to the symmetry of the second order differential operator

$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0, \quad \partial = \frac{d}{dt}, \tag{1.2}$$

with respect to the weight matrix W (see [D1]). The symmetry of D with respect to W is defined as

$$\int (PD) dW Q^* = \int P dW (QD)^*, \tag{1.3}$$

for any matrix polynomials P and Q. Here and in the rest of the paper we will follow the notation in [GT] for right-hand side differential operators. In particular, if P is a matrix polynomial and D a differential operator as (1.2), by PD we mean

$$PD = P''(t)F_2(t) + P'(t)F_1(t) + P(t)F_0.$$

As more families of orthogonal matrix polynomials satisfying second order differential equations become available many new interesting phenomena are being discovered, which are absent in the well known scalar theory.

One of such phenomena is the fact that the elements of a family of orthogonal matrix polynomials $(P_n)_n$ can be common eigenfunctions of several linearly independent second order differential operators (while in the scalar case the symmetric second order differential operator is unique up to multiplicative and additive constants). The first illustrations of this phenomenon have been recently found by F. A. Grünbaum and M. M. Castro [CG2] and other authors contributed more later. Some of the examples arise from group representation theory. For instance, [GPT1] discusses two second order differential operators acting on matrix spherical functions which were later put in the framework of orthogonal polynomials in [GPT2, GPT4, PT, PR]; see also [GdI]. Other examples were found by integrating an appropriate set of differential equations (see [D2, DdI, DL]). See also [CG1], where the authors take up the issue of existence of orthogonal matrix polynomials which are common eigenfunctions of differential operators of order one.

As a consequence of this phenomenon the algebra of differential operators associated with a fixed weight matrix W is receiving a lot of attention. This algebra $\mathcal{D}(W)$ is defined as follows: given a fixed sequence of orthogonal polynomials $(P_n)_n$ with respect to W (the monic sequence, for instance), $\mathcal{D}(W)$ is formed by all differential operators

$$D = \sum_{i=0}^{k} \partial^i F_i(t), \quad \partial = \frac{d}{dt}, \quad k \geq 0, \tag{1.4}$$

where $F_i(t), i = 0, \ldots, k$ are matrix polynomials of $\deg(F_i) \leq i$, for which $P_n D = \Gamma_n P_n$, $n \geq 0$. For the classical families of the scalar case every differential operator having one of the families as eigenfunctions has to be a polynomial in the corresponding symmetric second order differential operator (see [M]). Hence the associated algebra is isomorphic to $\mathbb{C}[t]$. The examples of weight matrices having several linearly independent symmetric second order differential operators show that in the matrix case the problem of characterizing the algebra $\mathcal{D}(W)$ is going to be a rather more difficult problem. In [CG2, GdI, DdI] some conjectures have been made about the structure of the algebra $\mathcal{D}(W)$ for some concrete examples. Based on computational evidence these conjectures show that we can expect a big variety of situations in the matrix case. To the best of our knowledge, only one of those conjectures has been proved for the weight matrix (1.6) (see [CG2], Sect. 6, for the conjecture and [T] for the proof).

The purpose of this paper is to show what one can call the dual situation to that described in the previous paragraph. For a fixed differential operator D of the form (1.4), we define a set of weight matrices

$$\Upsilon(D) = \{W : D \text{ is symmetric with respect to } W\}, \tag{1.5}$$

where the symmetry of D is defined again by (1.3).

Note that if $\Upsilon(D) \neq \emptyset$ then it is a convex cone: if $W_1, W_2 \in \Upsilon(D)$ and $\gamma, \zeta \geq 0$ (one of them non null), then $\gamma W_1 + \zeta W_2 \in \Upsilon(D)$.

The weight matrices W going along with a symmetric second order differential operator D mentioned at the beginning of this paper provide examples where $\Upsilon(D) \neq \emptyset$. In these examples $\Upsilon(D)$ contains at least a half line: γW, $\gamma > 0$. In this paper we show the first examples of operators D for which $\Upsilon(D)$ is a two dimensional convex cone. That is, we show examples of a fixed second order differential operator D as (1.2) for which there exist two weight matrices W_1 and W_2, $W_1 \neq \alpha W_2$ for any $\alpha > 0$, such that D is symmetric with respect to any of the weight matrices $\gamma W_1 + \zeta W_2$, $\gamma, \zeta \geq 0$. That means, in particular, that the corresponding monic matrix polynomials $(P_{n,\zeta/\gamma})_n$ orthogonal with respect to $\gamma W_1 + \zeta W_2$ (they only depend on W_1, W_2 and the ratio ζ/γ) are eigenfunctions of D

$$P_{n,\zeta/\gamma} D = \Gamma_n P_{n,\zeta/\gamma}, \quad n = 0, 1, \ldots, \quad \gamma > 0, \quad \zeta \geq 0,$$

where D and Γ_n do not depend on γ, ζ.

We give a simple but fruitful method to find such examples (Section 2) and show a collection of instructive examples (Section 3).

Our method itself is a surprise if one compares it with the situation in the scalar case. We first take a weight matrix W which has several linearly independent symmetric second order differential operators. And then we add a Dirac distribution $\delta_{t_0}M(t_0)$ to W, where the real number t_0 and the mass $M(t_0)$ (a Hermitian positive semidefinite matrix) are carefully chosen. We show in Section 2 that for a fixed t_0 and under certain mild conditions, we can produce a positive semidefinite matrix $M(t_0)$ and a second order differential operator D symmetric with respect to W, such that D is also symmetric with respect to any weight matrix of the form $\gamma W + \zeta\delta_{t_0}M(t_0)$, $\gamma > 0, \zeta \geq 0$. In Section 3 we illustrate with some examples that the choice of the point t_0 where the Dirac distribution is located depends more on the matrices $F_2(t_0)$, $F_1(t_0)$ and F_0 than on the support of W. We also characterize for some of these examples the convex cone (1.5).

The situation is quite different from the scalar case. When a mass point is added to any of the classical weights of Hermite, Laguerre and Jacobi, the existence of a symmetric second order differential operator automatically disappears. Only when t_0 is taken at the endpoints of the support one eventually gets the symmetry of a fourth (or even larger) order differential operator which is not symmetric with respect to the original weight. This arises for the particular cases of the Laguerre weight e^{-t} in $(0, +\infty)$, for the Legendre weight 1 in $(-1, 1)$ and for the special case of the Jacobi weight $(1-t)^\alpha$ in $(0, 1)$, raising the so called Laguerre type weight $e^{-t} + M\delta_0$, Legendre type weight $1 + M(\delta_{-1} + \delta_1)$ and Jacobi type weight $(1-t)^\alpha + M\delta_0$, respectively (see [LK], or [GH] where these and some other examples are obtained by applying the Darboux process).

We would like to conclude this Introduction by displaying one of our examples. Consider the weight matrix

$$W_a(t) = e^{-t^2}\begin{pmatrix} 1+a^2t^2 & at \\ at & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad a \in \mathbb{R}\setminus\{0\}. \tag{1.6}$$

The linear space of differential operators of order at most two having the orthogonal polynomials with respect to W_a as eigenfunctions has dimension five. A basis is formed by the identity and four linearly independent operators of order two (see Section 6 of [CG2]). We show in Section 3.1 that the weight matrices $W_{a,\gamma,\zeta} = \gamma W_a + \zeta\delta_0\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\gamma > 0, \zeta \geq 0$, share the following symmetric second order differential operator:

$$D_a = \partial^2\begin{pmatrix} 1-at & -1+a^2t^2 \\ -1 & 1+at \end{pmatrix} + \partial^1\begin{pmatrix} -2a-2t & 2a+2(2+a^2)t \\ 0 & -2t \end{pmatrix} + \partial^0\begin{pmatrix} -1 & 2\dfrac{2+a^2}{a^2} \\ \dfrac{4}{a^2} & 1 \end{pmatrix}.$$

Moreover, we prove that

$$\Upsilon(D_a) = \left\{\gamma W_a + \zeta\delta_0\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : \gamma > 0, \zeta \geq 0\right\}.$$

As we pointed out above, this means that each monic family $(P_{n,a,\zeta/\gamma})_n$, $\gamma > 0, \zeta \geq 0$, orthogonal with respect to $W_{a,\gamma,\zeta}$ satisfies the same second order differential equation, namely

$$P_{n,a,\zeta/\gamma}D_a = \Gamma_{n,a}P_{n,a,\zeta/\gamma}, \quad n = 0, 1, \ldots,$$

where

$$\Gamma_{n,a} = \begin{pmatrix} -(2n+1) & \dfrac{(2+na^2)(2+(n+1)a^2)}{a^2} \\ \dfrac{4}{a^2} & -2n+1 \end{pmatrix}.$$

Notice that neither D_a nor $\Gamma_{n,a}$ depend on γ, ζ.

We will also see in Section 3.1 that the point t_0 where the discrete mass is added can be located at any real number. There always exists a Hermitian positive semidefinite matrix $M(t_0)$ such that the weight matrices $\gamma W_a + \zeta\delta_{t_0}M(t_0)$, $\gamma > 0, \zeta \geq 0$, share a common symmetric second order differential operator (see (3.3)).

2 The main result

Presented in this section is a set of constraints to guarantee the symmetry of a differential operator of any order with respect to a weight matrix modified by adding a Dirac distribution at an arbitrary point. Before that we need some definitions and previous results.

We call an $N \times N$ matrix of measures W (supported in the real line) a weight matrix if it satisfies the following: the numerical matrix $W(\Omega)$ is positive semidefinite for any Borel set Ω; the integral $\mu_n = \int t^n dW(t)$ (called the n-th moment of W) exists and is finite for any $n \in \mathbb{N}$; and $\int P(t)\, dW(t)\, P^*(t)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular.

A Hermitian sesquilinear form in the linear space of matrix polynomials can be associated with a weight matrix W:

$$\langle P, Q \rangle = \int P(t)\, dW(t)\, Q^*(t),$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$.

We can now produce a sequence of orthogonal matrix polynomials $(P_n)_n$ with $\deg P_n = n$ and nonsingular leading coefficient such that $\langle P_n, P_m \rangle = \Delta_n \delta_{n,m}$ with Δ_n positive definite. If $\Delta_n = I$ we say that the sequence $(P_n)_n$ is orthonormal (for a much more complete introduction to matrix orthogonality, see [DG3] and references therein).

If one is considering possible applications of orthogonal matrix polynomials, it is natural to concentrate on the cases where some additional property holds. For instance, in [D1] one of the authors raised a problem of characterizing weight matrices whose orthonormal matrix polynomials are common eigenfunctions of some symmetric right-hand side second order differential operator D as (1.2) with Hermitian eigenvalues. We say that a differential operator D is symmetric with respect to a weight matrix W if

$$\langle PD, Q \rangle = \langle P, QD \rangle$$

for any pair of matrix polynomials P and Q. Recall that we are following the notation in [GT] for right-hand side differential operators. We already mentioned in the Introduction that in the last few years a large class of families of weight matrices W has been found having symmetric second order differential operators as (1.2).

The condition of symmetry for the pair made up of a weight matrix W and a differential operator D_k of order k can be established in terms of a set of difference and differential equations relating W and the coefficients of D_k. Indeed, if we write the right-hand side differential operator D_k of order k as

$$D_k = \sum_{i=0}^{k} \partial^i F_i(t), \quad \partial = \frac{d}{dt}, \tag{2.1}$$

where $F_i(t), i = 0, \ldots, k$ are matrix polynomials of degree less than or equal to i,

$$F_i(t) = \sum_{j=0}^{i} t^j F_j^i, \quad F_j^i \in \mathbb{C}^{N\times N},$$

and denote by μ_n, $n = 0, 1, \ldots$, the moments of the weight matrix W, then we have the following:

Theorem 2.1. *For a weight matrix W the following two conditions are equivalent:*

1. *The operator D_k is symmetric with respect to W.*
2. *For $n \geq l$, the following $k+1$ sets of moment equations hold*

$$\sum_{i=0}^{k-l} \binom{k-i}{l} (n-l)_{k-l-i} B_n^{k-i} = (-1)^l (B_n^l)^*, \quad l = 0, \ldots, k, \tag{2.2}$$

where

$$B_n^l = \sum_{i=0}^{l} F_{l-i}^l \mu_{n-i}, \quad l = 0, \ldots, k.$$

Moreover, suppose the weight matrix $W = W(t)dt$ has a smooth density $W(t)$ with respect to the Lebesgue measure which satisfies the boundary conditions that

$$\sum_{i=0}^{p-1} (-1)^{k-i+p-1} \binom{k-i}{l} \left(F_{k-i} \cdot W\right)^{(p-1-i)}, \quad p = 1, \ldots, k, \quad l = 0, \ldots, k-p, \tag{2.3}$$

should have vanishing limits at each of the endpoints of the support of W, and the following $k+1$ matrix differential equations hold

$$\sum_{i=0}^{k-l} (-1)^{k-i} \binom{k-i}{l} \left(F_{k-i} \cdot W\right)^{(k-i-l)} = W \cdot F_l^*, \quad l = 0, \ldots, k. \tag{2.4}$$

Then the differential operator D_k (defined in (2.1)) is symmetric with respect to the weight matrix W.

(In (2.2) we are using the notation for the falling or bounded factorial $(x)_n$ defined by $(x)_n = x(x-1)\cdots(x-n+1) \quad \text{for} \quad n > 0,\ (x)_0 = 1.$)

Proof. The first and second parts are shown to be equivalent in Proposition 4 in [DdI]. Likewise, the last part can be found in Theorem 5 in [DdI].

□

Two advantages of the moment equations (2.2) are that they are equivalent to the symmetry of D_k and that there are no additional assumptions on the weight matrix W (in particular, neither smoothness nor boundary conditions are required). However, even if one can solve these moment equations it can be very difficult to recover the weight matrix W from its moments. Besides that the moment equations turn out to become a suitable tool in all the examples throughout this paper since those equations are going to be the key to characterize the convex cone $\Upsilon(D)$ associated with some of the differential operators D considered in the next section. The moment and symmetry equations for differential operators of order two appeared for the first time in [D1] when $F_0W = WF_0^*$ and later in [DG1] (see also [GPT3] in the case of the symmetry equations). As we remarked in the Introduction, most of the examples which appeared in the last years have been obtained solving these equations, while some others came from group representation theory.

We now present a result that will be used to generate examples of second order differential operators having infinitely many families of orthogonal matrix polynomials as eigenfunctions. The idea is to find certain constraints which guarantee the symmetry of a differential operator with respect to both W and a new weight matrix obtained from W by adding a Dirac distribution at one point.

Let W be a weight matrix and consider

$$\widetilde{W}(t) = W(t) + \delta_{t_0}(t)M(t_0), \tag{2.5}$$

where $\delta_{t_0}(t) = \delta(t-t_0)$ is the Dirac delta distribution or the "impulse symbol" introduced by P. A. M. Dirac in [D], which we will consider as a measure. The Hermitian positive semidefinite matrix $M(t_0)$ depends on the point where the Dirac distribution is added.

Weight matrices of the form (2.5) were considered in [YMP1, YMP2] (to study asymptotic properties of the corresponding modified Jacobi matrix) for W in the Nevai class, i.e., with convergent recurrence coefficients.

The moments of $\widetilde{W}$ are related with the moments of W by the formula

$$\widetilde{\mu}_n = \int t^n d\widetilde{W}(t) = \mu_n + \int t^n \delta_{t_0}(t)M(t_0)dt = \mu_n + t_0^n M(t_0), \quad n = 0, 1, \ldots$$

Observe that in the special case of $t_0 = 0$ the only modified moment is the first one $\widetilde{\mu}_0 = \mu_0 + M(0)$, and then $\widetilde{\mu}_n = \mu_n$ for $n = 1, 2, \ldots$

The following theorem gives conditions for the symmetry of a differential operator D_k with respect to the weight matrices W and $W + \delta_{t_0}M(t_0)$.

Theorem 2.2. *Let D_k be a differential operator of order k as in (2.1). Let W be a weight matrix. Assume that associated with the real point $t_0 \in \mathbb{R}$ there exists a Hermitian positive semidefinite matrix $M(t_0)$ satisfying*

$$
\begin{aligned}
F_j(t_0)M(t_0) &= 0, \quad j = 1, \dots, k, \\
F_0 M(t_0) &= M(t_0)F_0^*.
\end{aligned}
\tag{2.6}
$$

Then the operator D_k is symmetric with respect to W if and only if it is symmetric with respect to $\widetilde{W} = W + \delta_{t_0} M(t_0)$.

Proof. Recalling definitions around (2.2) for $\widetilde{W} = W + \delta_{t_0} M(t_0)$, we produce

$$
\widetilde{B}_n^l = \sum_{i=0}^{l} F_{l-i}^l \widetilde{\mu}_{n-i} = B_n^l + t_0^{n-l} F_l(t_0) M(t_0), \quad l = 0, \dots, k.
$$

Using conditions (2.6) for $j = 1, \dots, k$, we obtain

$$
\widetilde{B}_n^0 = B_n^0 + t_0^n F_0 M(t_0), \quad \widetilde{B}_n^l = B_n^l, \quad l = 1, \dots, k.
$$

Consequently, this shows that equations (2.2), $l = 1, \dots, k$, are just the same for W and $\widetilde{W}$. For $l = 0$, equations (2.2) for W and $\widetilde{W}$ are, respectively:

$$
\sum_{i=0}^{k-1} (n)_{k-i} B_n^{k-i} + B_n^0 = (B_n^0)^*,
$$

$$
\sum_{i=0}^{k-1} (n)_{k-i} B_n^{k-i} + B_n^0 + t_0^n F_0 M(t_0) = (B_n^0)^* + t_0^n M(t_0) F_0^*.
$$

The last condition in (2.6) shows again that those equations are the same for W and $\widetilde{W}$.
□

Note that for $N = 1$ (i.e. in the scalar case) Theorem 2.2 implies that either $M = 0$ or there exists a common zero for all coefficients of the differential operator. For instance, for $k = 2$ there is no such a common zero for the classical families of Hermite, Laguerre and Jacobi.

For a weight matrix W the constraints (2.6) mean that by adding a Dirac distribution to W the chances of W and $W + \delta_{t_0} M(t_0)$ sharing a symmetric differential operator of order k increase with the number of linearly independent symmetric differential operators of order k for W. In fact, all the examples we show in the next section are built from a weight matrix W having several linearly independent second order differential operators.

Let us notice that once we generate W, D_k, t_0 and $M(t_0)$ satisfying constraints (2.6) of the theorem above we can produce not only one single weight matrix for which D_k is symmetric, but also a two dimensional convex cone of weight matrices for which D_k is symmetric as well. If Theorem 2.2 holds for $W + \delta_{t_0} M(t_0)$ then automatically also holds for $\gamma W + \zeta \delta_{t_0} M(t_0)$

where $\gamma > 0$ and $\zeta \geq 0$. All the examples of weight matrices in $\Upsilon(D_k)$ which we show in this paper differ (up to a multiplicative constant) in a Dirac distribution. We think that this is by no means a general result, and are confident that other different situations may occur (such as the existence of differential operators D_k for which $\Upsilon(D_k)$ contains a two parametric family of weight matrices absolutely continuous with respect to the Lebesgue measure).

We will characterize the convex cone $\Upsilon(D)$ for some of the examples in the next section. To do that we will need the following result (which it is interesting in its own right). Consider the Fourier transform $\mathcal{F}(W)$ of a weight matrix W defined by

$$\mathcal{F}(W)(x) = \int_{\mathbb{R}} e^{itx} dW(t), \quad x \in \mathbb{R}.$$

Lemma 2.1. *Assume that the weight matrix W has moments $(\mu_n)_n$ satisfying*

$$\lim_m \operatorname{Tr}(\mu_{2m}) \frac{r^{2m}}{(2m)!} = 0, \quad \textit{for all} \quad 0 < r < R, \quad R > 0, \tag{2.7}$$

where $\operatorname{Tr}(X)$ stands for the trace of the matrix X. Then there exists an analytic function Φ in the strip $\{z \in \mathbb{C} : |\Im z| < R\}$ such that $\Phi(x) = \mathcal{F}(W)(x)$, $x \in \mathbb{R}$.

Proof. For any positive semidefinite matrix X it is straightforward that $0 \preccurlyeq X \preccurlyeq \operatorname{Tr}(X)I$ (where $\preccurlyeq$ stands for the usual positive semidefinite ordering). Using the Cauchy-Schwarz inequality we have that

$$|uXv^*| \leq |uX^{1/2}X^{1/2}v^*| \leq (uXu^*)^{1/2}(vXv^*)^{1/2} \leq \|u\|\|v\| \operatorname{Tr}(X),$$

for any vectors $u, v \in \mathbb{C}^N$ (where $\|u\| = \sqrt{uu^*}$).

For a weight matrix W ($\operatorname{Tr}(W)$ is a positive measure), the previous inequality gives

$$|uWv^*| \leq \|u\|\|v\| \operatorname{Tr}(W), \tag{2.8}$$

for any vectors $u, v \in \mathbb{C}^N$.

The Fourier transform $\mathcal{F}(W)$ of a weight matrix W is a $\mathcal{C}^\infty$ function in $\mathbb{R}$ and

$$\mathcal{F}(W)^{(n)}(x) = \int_{\mathbb{R}} (it)^n e^{itx} dW(t), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{2.9}$$

Fixing a number $a \in \mathbb{R}$, the Lagrange remainder for the Taylor formula gives for each $x \in \mathbb{R}$ and $k \geq 0$ a real number $y_{k,x}$ for which

$$\mathcal{F}(W)(x) = \sum_{n=0}^{k-1} \mathcal{F}(W)^{(n)}(a) \frac{(x-a)^n}{n!} + \mathcal{F}(W)^{(k)}(y_{k,x}) \frac{(x-a)^k}{k!}.$$

Assume that we have proved that for all $x \in \mathbb{R}$, $|x - a| < R$,

$$\lim_k \mathcal{F}(W)^{(k)}(y_{k,x}) \frac{(x-a)^k}{k!} = 0. \tag{2.10}$$

The power series

$$\sum_{n=0}^{\infty} \mathcal{F}(W)^{(n)}(a)\frac{(x-a)^n}{n!}$$

defines an analytic function Φ_a in $\{z \in \mathbb{C} : |z-a| < R\}$ satisfying $\Phi_a(x) = \mathcal{F}(W)(x)$, $x \in \mathbb{R}$, $|x-a| < R$. The Lemma now follows easily by using a standard process of analytic continuation.

Let us now prove (2.10). This is equivalent to proving that for any vectors $u, v \in \mathbb{C}^N$

$$\lim_k u\mathcal{F}(W)^{(k)}(y_{k,x})v^* \frac{(x-a)^k}{k!} = 0, \quad |x-a| < R. \tag{2.11}$$

Using (2.9) and (2.8) we have

$$\begin{aligned} |u\mathcal{F}(W)^{(k)}(y_{k,x})v^*| &= \left| \int_{\mathbb{R}} (it)^k e^{ity_{k,x}} u dW(t)v^* \right| \\ &\leq \int_{\mathbb{R}} |t|^k |udW(t)v^*| \leq \|u\|\|v\| \int_{\mathbb{R}} |t|^k d\,\mathrm{Tr}(W)(t). \end{aligned}$$

Hence if k is even

$$|u\mathcal{F}(W)^{(k)}(y_{k,x})v^*| \leq \|u\|\|v\|\,\mathrm{Tr}(\mu_k);$$

and if k is odd then

$$|u\mathcal{F}(W)^{(k)}(y_{k,x})v^*| \leq \|u\|\|v\|(\mathrm{Tr}(\mu_0) + \mathrm{Tr}(\mu_{k+1})).$$

The limit (2.11) follows now from (2.7).

□

For a fixed differential operator D of the form (1.4), we can associate another set of weight matrices

$$\mathfrak{X}(D) = \{W : P_n^W D = \Gamma_n P_n^W, \quad n \geq 0\},$$

where $(P_n^W)_n$ is the sequence of monic polynomials orthogonal with respect to W.

Note that $\Upsilon(D) \subset \mathfrak{X}(D)$ and if $\mathfrak{X}(D) \neq \emptyset$ then it is a cone: if $W \in \mathfrak{X}(D)$ then $\alpha W \in \mathfrak{X}(D)$ for any $\alpha > 0$.

In general we have $\Upsilon(D) \neq \mathfrak{X}(D)$, as the following example shows. Let D be a symmetric second order differential operator with respect to a certain weight matrix W. As a consequence the monic polynomials orthogonal with respect to W are eigenfunctions for D. It is clear that iD is not symmetric with respect to W but the monic polynomials orthogonal with respect to W are still eigenfunctions for iD. That means that $W \notin \Upsilon(iD)$ but $W \in \mathfrak{X}(iD)$, and then $\Upsilon(iD) \neq \mathfrak{X}(iD)$.

We are concerned that some natural questions arise regarding the relationship between $\Upsilon(D)$ and $\mathfrak{X}(D)$, but they are out of the scope of this paper. We would like to quote the concluding remark in the Introduction of [CG2] because it suits very well with this situation: "We emphasize something that will be apparent to any reader of this paper: the full picture of the phenomenon in question is still far from being complete. This paper is an attempt to describe clearly some of the new problems that one faces in the matrix-valued case, and we pick a few examples that should give an idea of the richness of the situation at hand".

3 Examples

In this section we exhibit a collection of instructive examples. The first three examples of (2×2) weight matrices W (supported in $(-\infty, +\infty)$, $(0, +\infty)$ and $(0, 1)$, respectively) have the property that they provide four linearly independent symmetric second order differential operators having a fixed family of orthogonal matrix polynomials with respect to W as eigenfunctions. We show that for any real number t_0 we can find a positive semidefinite matrix $M(t_0)$ and a symmetric second order differential operator D as in (1.2) satisfying the constraints (2.6). According to Theorem 2.2, the operator D will be also symmetric with respect to $\gamma W + \zeta\delta_{t_0} M(t_0)$, $\gamma > 0, \zeta \geq 0$.

The last example deals with a $(N \times N)$ weight matrix W (supported in $(0, +\infty)$) with at least two linearly independent symmetric second order differential operators. In this case we locate t_0 at 0, i.e. one of the endpoints of the support of W, with a mass M carefully chosen.

For simplicity, our selection of examples is restricted to the field of real numbers and to matrices with real entries, but the method we use to find these examples is not restricted to this case.

More examples appear in the PhD dissertation of one of the authors [dI].

3.1 $W_a(t) = e^{-t^2} e^{At} e^{A^*t}$ with $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $a \in \mathbb{R} \setminus \{0\}$

This weight matrix was introduced for the first time in Section 5.1 in [DG1] (for arbitrary size $N \times N$). It was deeply explored in [DG2] and a set of generators of second order differential operators can be found in Section 6 in [CG2].

By expanding the exponential, we find that

$$W_a(t) = e^{-t^2} e^{At} e^{A^*t} = e^{-t^2} \begin{pmatrix} 1 + a^2t^2 & at \\ at & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\}. \tag{3.1}$$

We need an expression for the (real) linear space of symmetric differential operators of order at most two with respect to W_a. To do that we solve equations (2.4) for $k = 2$. For the benefit of the reader, we recall here these equations

$$\begin{aligned} F_2 W &= W F_2^*, \\ 2(F_2 W)' &= F_1 W + W F_1^*, \\ (F_2 W)'' - (F_1 W)' + F_0 W &= W F_0^*. \end{aligned} \tag{3.2}$$

We then get an expression for the 5-dimensional (real) linear space of symmetric differential operators of order at most two with respect to W_a. Then, for a fixed real number t_0, we solve the equations (2.6). In this case, we find that the following differential operator

$$D_{a,t_0} = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0, \tag{3.3}$$

where

$$F_2(t) = \begin{pmatrix} -\xi^{\mp}_{a,t_0} + at_0 - at & -1 - (a^2 t_0)t + a^2 t^2 \\ -1 & -\xi^{\mp}_{a,t_0} + at \end{pmatrix},$$

$$F_1(t) = \begin{pmatrix} -2a + 2\xi^{\mp}_{a,t_0} t & -2t_0 - 2a\xi^{\mp}_{a,t_0} + 2(2+a^2)t \\ 2t_0 & 2(\xi^{\mp}_{a,t_0} - at_0)t \end{pmatrix},$$

$$F_0 = \begin{pmatrix} \xi^{\mp}_{a,t_0} + 2\dfrac{t_0}{a} & 2\dfrac{2+a^2}{a^2} \\ \dfrac{4}{a^2} & -\xi^{\mp}_{a,t_0} - 2\dfrac{t_0}{a} \end{pmatrix},$$

and the Hermitian positive semidefinite matrix $M(t_0)$

$$M(t_0) = M(a, t_0) = \begin{pmatrix} (\xi^{\pm}_{t_0,a})^2 & \xi^{\pm}_{t_0,a} \\ \xi^{\pm}_{t_0,a} & 1 \end{pmatrix},$$

where

$$\xi^{\pm}_{a,t_0} = \frac{at_0 \pm \sqrt{4 + a^2 t_0^2}}{2},$$

satisfy the constraints (2.6).

This differential operator can be obtained as a linear combination of the second order differential operators introduced in [CG2] (D_i, $i = 1, 2, 3, 4$), namely

$$D_{a,t_0} = \Big(-\xi^{\mp}_{a,t_0} + \frac{2t_0}{a} \Big) I - \xi^{\mp}_{a,t_0} D_1 - \frac{4t_0}{a} D_2 + \frac{4}{a^2} D_4.$$

Using $\xi^{+}_{a,t_0}\xi^{-}_{a,t_0} + 1 = 0$ it is easy to verify that the coefficients of D_{a,t_0} evaluated at t_0 satisfy the conditions (2.6). Since D_{a,t_0} has been chosen to be symmetric with respect to the weight matrix (3.1), Theorem 2.2 implies that D_{a,t_0} is also symmetric with respect to any of the following weight matrices:

$$W_{a,t_0,\gamma,\zeta}(t) = \gamma W_a(t) + \zeta \delta_{t_0}(t) M(a, t_0), \quad \gamma > 0, \zeta \geq 0.$$

We now prove that our method provides all the weight matrices in the convex cone $\Upsilon(D_{a,t_0})$ defined in (1.5), that is

$$\Upsilon(D_{a,t_0}) = \{\gamma W_a(t) + \zeta \delta_{t_0}(t) M(a, t_0); \quad \gamma > 0, \zeta \geq 0\}.$$

Theorem 2.1 implies that the sequence of moments $(\mu_n)_n$ of each weight matrix $U \in \Upsilon(D)$ has to satisfy the moment equations (2.2) for $k = 2$ (to simplify the notation we remove the dependence on a and t_0). These moment equations are

$$F_2^2\mu_n + F_1^2\mu_{n-1} + F_0^2\mu_{n-2} - \mu_n(F_2^2)^* - \mu_{n-1}(F_1^2)^* - \mu_{n-2}(F_0^2)^* = 0, \quad n \geq 2; \tag{3.4}$$

$$2(n-1)(F_2^2\mu_n + F_1^2\mu_{n-1} + F_0^2\mu_{n-2}) + (F_1^1\mu_n + F_0^1\mu_{n-1}) + \mu_n(F_1^1)^* + \mu_{n-1}(F_0^1)^* = 0, \quad n \geq 1; \tag{3.5}$$

and

$$n(n-1)(F_2^2\mu_n + F_1^2\mu_{n-1} + F_0^2\mu_{n-2}) + n(F_1^1\mu_n + F_0^1\mu_{n-1}) + F_0\mu_n - \mu_n(F_0)^* = 0, \quad n \geq 0, \tag{3.6}$$

where F_2^2, F_1^2, F_0^2, F_1^1, F_0^1 and F_0 are, respectively, the coefficients of the polynomials F_2, F_1 and F_0. Let us recall that $D = \partial^2 F_2 + \partial F_1 + \partial^0 F_0$.

From equations (3.4) and (3.5), we get the following expression:

$$\begin{aligned}((n-1)F_2^2 + F_1^1)\mu_n + \mu_n((n-1)F_2^2 + F_1^1)^* = ((1-n)F_1^2 - F_0^1)\mu_{n-1} + \mu_{n-1}((1-n)F_1^2 - F_0^1)^* \\ + (1-n)(F_0^2\mu_{n-2} + \mu_{n-2}(F_0^2)^*).\end{aligned} \tag{3.7}$$

In our example we have that

$$(n-1)F_2^2 + F_1^1 = \begin{pmatrix} 2\xi_{a,t_0}^{\mp} & (n-1)a^2 + 2(2+a^2) \\ 0 & -2\xi_{a,t_0}^{\pm} \end{pmatrix}.$$

For a fixed $n \geq 1$ the matrices $(n-1)F_2^2 + F_1^1$ and $-((n-1)F_2^2 + F_1^1)^*$ do not share any eigenvalue. This implies that equation (3.7) defines μ_n, $n \geq 1$, in a unique way given μ_0 (see [Ga], p. 225).

Equation (3.6) for $n = 0$ implies that μ_0 has to satisfy

$$F_0\mu_0 = \mu_0(F_0)^*.$$

It is just a matter of computation to see that the set of solutions μ_0 of the previous equation is formed by the first moment of the weight matrices $W_{\gamma,\zeta} = \gamma W + \zeta\delta_{t_0}M$, $\gamma, \zeta \in \mathbb{R}$. Hence, each weight matrix $U \in \Upsilon(D)$ has, for certain $\gamma, \zeta \in \mathbb{R}$, the same moments $(\mu_n)_n$ as $\gamma W + \zeta\delta_{t_0}M$. From the expression (3.1) for $W_{\gamma,\zeta}$, we deduce that for $m = 0, 1, \ldots,$

$$\mu_{2m} = \gamma\begin{pmatrix} h_{2m} + a^2 h_{2m+2} & 0 \\ 0 & h_{2m} \end{pmatrix} + \zeta t_0^{2m} M(t_0), \tag{3.8}$$

$$\mu_{2m+1} = \gamma\begin{pmatrix} 0 & ah_{2m+2} \\ ah_{2m+2} & 0 \end{pmatrix} + \zeta t_0^{2m+1} M(t_0), \tag{3.9}$$

where

$$h_{2m} = \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2m)!}{4^m m!}, \quad h_{2m+1} = 0,$$

are the Hermite moments.

Since the moment μ_0 has to be positive definite and $M(t_0)$ is singular, we deduce that $\gamma > 0$.

We now prove that $U = \gamma W + \zeta\delta_{t_0} M$ and consequently $\zeta \geq 0$. This can be done using different approaches. One of them is via Fourier transform and it works as follows.

Equations (3.8) and (3.9) show that for any $R > 0$,

$$\lim_m \mathrm{Tr}(\mu_{2m}) \frac{R^{2m}}{(2m)!} = 0.$$

According to Lemma 2.1, there exists an entire function Φ such that $\Phi(x) = \mathcal{F}(U)(x)$, $x \in \mathbb{R}$. From (3.1) the Fourier transform of $\mathcal{F}(W_{\gamma,\zeta})$ can be computed explicitly ([Le], (4.11.4)):

$$\mathcal{F}(W_{\gamma,\zeta})(x) = \gamma\sqrt{\pi}e^{-x^2/4}\begin{pmatrix} 1 + a^2(2 - x^2)/4 & ix/2 \\ ix/2 & 1 \end{pmatrix} + \zeta e^{it_0 x} M.$$

This shows that $\mathcal{F}(W_{\gamma,\zeta})$ is actually an entire function.

From (2.9) one can see that the moments $(\mu_n)_n$ of a weight matrix W are, up to a multiplicative constant, the derivatives at 0 of its Fourier transform: $\mu_n = \mathcal{F}(W)^{(n)}(0)/i^n$, $n \geq 0$. Since U and $W_{\gamma,\zeta}$ have the same moments, we conclude that the corresponding Fourier transforms have at 0 the same derivatives of any order. That is, the entire functions Φ and $\mathcal{F}(W_{\gamma,\zeta})$ are equal and then $\mathcal{F}(W_{\gamma,\zeta})(x) = \mathcal{F}(U)(x)$, $x \in \mathbb{R}$. So $U = W_{\gamma,\zeta}$, in which case ζ has to be bigger than or equal to 0 (since U is a weight matrix).

3.2 $W_{a,\alpha}(t) = t^\alpha e^{-t} t^B t^{B^*}$ with $B = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$, $a \in \mathbb{R} \setminus \{0\}$

This weight matrix was introduced for the first time in Section 6.2 in [DG1] (for arbitrary size $N \times N$) and it was extensively studied in [DL]. Unlike the first example, its algebra of differential operators has not been studied in depth, but as we mentioned at the beginning of this section, by solving equations (3.2), one finds that there are four linearly independent symmetric second order differential operators.

By computing t^B for $B = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ we find that

(3.10)

$$W_{a,\alpha}(t) = t^\alpha e^{-t} t^B t^{B^*} = t^\alpha e^{-t}\begin{pmatrix} t^2 + a^2(t-1)^2 & a(t-1) \\ a(t-1) & 1 \end{pmatrix}, \quad t \in (0, +\infty), \quad \alpha > -1.$$

Proceeding as in Section 3.1 for a fixed real number t_0, we find that the following differential operator (symmetric with respect to $W_{a,\alpha}$)

$$D_{a,\alpha,t_0} = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0,$$

where

$$F_2(t) = \begin{pmatrix} at_0 & a^2t_0 \\ -t_0 & -at_0 \end{pmatrix} + t \begin{pmatrix} \phi^\pm - \frac{1+(\alpha+t_0)(1+a^2)}{a} & -(\alpha+t_0+1)(1+a^2) \\ 0 & \phi^\pm + a \end{pmatrix} + t^2 \begin{pmatrix} 0 & (1+a^2)(1+\alpha) \\ 0 & 0 \end{pmatrix},$$

$$F_1(t) = \begin{pmatrix} \begin{pmatrix} -a(3t_0 - 2 + (\alpha+1)^2) \\ +(\alpha+3)(\phi^\pm - \frac{t_0+\alpha+1}{a}) \end{pmatrix} & \begin{pmatrix} 2a\phi^\pm - (1+a^2)(\alpha^2+2t_0+3\alpha) \\ +\alpha t_0(a^2-1) - (t_0+\alpha+3) \end{pmatrix} \\ t_0 - \alpha - 1 & \phi^\pm(\alpha+1) - a\alpha t_0 \end{pmatrix} + t \begin{pmatrix} -\phi^\pm + a(t_0-1) + \frac{t_0+\alpha+1}{a} & (1+a^2)(\alpha^2+3\alpha-t_0\alpha+2) + 2(\alpha-1) \\ 0 & -\phi^\pm + a\alpha \end{pmatrix},$$

$$F_0 = \begin{pmatrix} -\frac{\phi^\pm}{2} + \frac{a(t_0-1)}{2} + \frac{(\alpha+2)(t_0+\alpha+1)}{2a} - \frac{a(1+\alpha)}{1+a^2} & 1 + a\alpha\phi^\pm - \alpha(t_0-1)(a^2+\alpha+2) + \frac{1+\alpha}{1+a^2} \\ \frac{1+\alpha}{1+a^2} & \frac{\phi^\pm}{2} - \frac{a(t_0-1)}{2} - \frac{(\alpha+2)(t_0+\alpha+1)}{2a} + \frac{a(1+\alpha)}{1+a^2} \end{pmatrix},$$

and the Hermitian positive semidefinite matrix $M(t_0)$

$$M(t_0) = M(a,\alpha,t_0) = \begin{pmatrix} (\phi^\pm_{a,\alpha,t_0})^2 & \phi^\pm_{a,\alpha,t_0} \\ \phi^\pm_{a,\alpha,t_0} & 1 \end{pmatrix},$$

where

$$\phi^\pm = \phi^\pm_{a,\alpha,t_0} = \frac{1}{2}\frac{(a^2+1)(t_0+\alpha) - a^2 + 1 \pm \sqrt{(a^2+1)(a^2(t_0-\alpha-1)^2 + (t_0+\alpha+1)^2)}}{a},$$

satisfy the constraints (2.6).

Thus, Theorem 2.2 implies that D_{a,α,t_0} is symmetric with respect to any of the following weight matrices:

$$W_{a,\alpha,t_0,\gamma,\zeta}(t) = \gamma W_{a,\alpha}(t)\chi_{(0,+\infty)}(t) + \zeta\delta_{t_0}(t)M(a,\alpha,t_0), \quad t \in \mathbb{R}, \quad \alpha > -1, \quad \gamma > 0, \zeta \geq 0.$$

Note that the discrete mass on the Delta distribution can be located in or out of the support of the original weight matrix (3.10).

Our method provides all the weight matrices in the convex cone $\Upsilon(D_{a,\alpha,t_0})$, i.e.

$$\Upsilon(D_{a,\alpha,t_0}) = \{W_{a,\alpha,t_0,\gamma,\zeta}(t); \quad \gamma > 0, \zeta \geq 0\}.$$

This can be proved in a similar way as for the previous example. We have that

$$(n-1)F_2^2 + F_1^1 = \begin{pmatrix} \phi^\mp - a\alpha & (n-1)(1+\alpha)(1+a^2) \\ 0 & -\phi^\pm + a\alpha \end{pmatrix},$$

hence, for a fixed $n \geq 1$ again the matrices $(n-1)F_2^2 + F_1^1$ and $-((n-1)F_2^2 + F_1^1)^*$ do not share any eigenvalue.

From the expression (3.10) for $W_{\gamma,\zeta}$ we deduce that for $n = 0, 1, \ldots$

$$\mu_n = \gamma\Gamma(n+\alpha+1)\begin{pmatrix} \theta_n & a(\alpha+n) \\ a(\alpha+n) & 1 \end{pmatrix} + \zeta t_0^n M(t_0) \tag{3.11}$$

where

$$\theta_n = a^2(\alpha^2 + (2n+1)\alpha + n^2 + n + 1) + \alpha^2 + (2n+3)\alpha + (n+1)(n+2),$$

and that $\mathcal{F}(W_{a,\alpha,t_0,\gamma,\zeta})$ is analytic in the cut plane $\mathbb{C} \setminus \{-ix : x \in \mathbb{R}, 1 < x\}$ (the explicit expression of $\mathcal{F}(W_{a,\alpha,t_0,\gamma,\zeta})$ can be computed easily from [Le], (1.5.1)).

Equation (3.11) shows that for all $0 < r < 1$

$$\lim_m \operatorname{Tr}(\mu_{2m})\frac{r^{2m}}{(2m)!} = 0.$$

And we can proceed as in the previous example applying Lemma 2.1 in the strip $\{z \in \mathbb{C} : |\Im z| < 1\}$.

3.3 An example supported in $(0, 1)$

The following weight matrix is a modification of the one introduced in [PT]:

$$W_{\alpha,\beta,k}(t) = t^\alpha(1-t)^\beta \begin{pmatrix} kt^2 + \beta - k + 1 & (\beta-k+1)(1-t) \\ (\beta-k+1)(1-t) & (\beta-k+1)(1-t)^2 \end{pmatrix}, \quad t \in (0,1),$$

where $\alpha, \beta > -1$ and $0 < k < \beta + 1$.

The difference between $W_{\alpha,\beta,k}$ and the weight matrix introduced in [PT] ($N = 2$), is that $W_{\alpha,\beta,k}$ enjoys four linearly independent symmetric second order differential operators while that in [PT] enjoys only two. The example in [PT] is related to a group-theoretical situation but this is not the case of our $W_{\alpha,\beta,k}$, as far as we know.

As in the previous examples, for a fixed real number t_0 (except for $t_0 = -\frac{1}{k}(\alpha + \beta - k + 2)$), no matter if it is located in or out of the support of the weight matrix $W_{\alpha,\beta,k}$, we find a symmetric second order differential operator D_{α,β,k,t_0} with respect to $W_{\alpha,\beta,k}$ and a Hermitian positive semidefinite matrix $M(t_0)$ satisfying the constrains (2.6). Hence, this operator D_{α,β,k,t_0} is also symmetric for any of the weight matrices $\gamma W_{\alpha,\beta,k}\chi_{(0,1)} + \zeta\delta_{t_0}M(t_0)$, $\gamma > 0, \zeta \geq 0$. The convex cone $\Upsilon(D_{a,\alpha,\beta,t_0})$ is formed by those weight matrices. This can be proved as in the previous examples. Since in this example the weight matrices have compact support $[0, 1] \cup \{t_0\}$, it follows that the moments $(\mu_n)_n$ satisfy $\operatorname{Tr}(\mu_n) \leq \gamma \operatorname{Tr}(\mu_0) + \operatorname{Tr}(M(t_0))|\zeta||t_0|^n$, $n \geq 0$. Then by Lemma 2.1 the Fourier transforms are entire functions.

Since the formulas for arbitrary α, β and k are very long we show here only one concrete example: $\alpha = 0$, $\beta = 0$ and $k = 1/2$:

$$F_2(t) = \begin{pmatrix} (1-t)(-t_0 + t(2t_0 - 3)) + \dfrac{t(1-t)(1-t_0)}{\varphi^{\pm}} & 2t + t_0 - 2t_0 t - t_0 t^2 \\ -t_0(1-t)^2 & (t-1)(t-t_0) + \dfrac{t(1-t)(1-t_0)}{\varphi^{\pm}} \end{pmatrix},$$

$$F_1(t) = \begin{pmatrix} 12t - 10 + 8t_0(1-t) + \dfrac{(t_0-1)(4t-3)}{\varphi^{\pm}} & 9 - t - 4t_0(t+2) + \dfrac{2(t_0-1)}{\varphi^{\pm}} \\ (1-4t_0)(t-1) & 4(t-t_0) + \dfrac{(t_0-1)(4t-1)}{\varphi^{\pm}} \end{pmatrix},$$

$$F_0 = \begin{pmatrix} \dfrac{(1-t_0)(1+2\varphi^{\pm})}{2\varphi^{\pm}} & -t_0 - 3 + \dfrac{1-t_0}{2\varphi^{\pm}} \\ -t_0 + \dfrac{1-t_0}{2\varphi^{\pm}} & -\dfrac{(1-t_0)(1+2\varphi^{\pm})}{2\varphi^{\pm}} \end{pmatrix},$$

and

$$M(t_0) = \begin{pmatrix} 1 & \varphi^{\pm} \\ \varphi^{\pm} & (\varphi^{\pm})^2 \end{pmatrix},$$

where

$$\varphi^{\pm}(t_0) = \frac{2 - t_0 \pm \sqrt{2t_0^2 - 2t_0 + 1}}{t_0 + 3}.$$

We need to impose $t_0 \neq -3$ to avoid singularities in $M(t_0)$ ($t_0 = 1$ gives $\varphi^-(1) = 0$, in which case $(1-t_0)/\varphi^-$ has to be taken equal to 1 in the entries of F_2, F_1 and F_0).

3.4 An example of arbitrary size

We consider here a weight matrix defined by

$$W(t) = t^{\alpha} e^{-t} e^{At} t^{\frac{1}{2}J} t^{\frac{1}{2}J^*} e^{A^* t}, \quad t \in (0, \infty), \quad \alpha > -1, \tag{3.12}$$

where

$$J = \sum_{i=1}^{N} (N-i) E_{ii}, \quad A = \sum_{i=1}^{N-1} \nu_i E_{i,i+1}, \quad \nu_i \in \mathbb{R} \setminus \{0\}. \quad i = 1, \dots, N-1. \tag{3.13}$$

Here we are using E_{ij} to denote the matrix with entry (i, j) equal to 1 and 0 otherwise.

This weight matrix was introduced for the first time in [DdI]. This example enjoys the special property of having symmetric odd order differential operators, a phenomenon that is not possible in the classical scalar theory. For more details, the reader should consult [DdI].

It is proved in [DdI] that (3.12) always has a symmetric second order differential operator given by

$$D_1 = \partial^2 t I + \partial^1 [(\alpha + 1) I + J + t(A - I)] + \partial^0 [(J + \alpha I) A - J],$$

where A and J are defined in (3.13). Note that there are $N-1$ free parameters in D_1. Assuming the following conditions on the parameters $\nu_1, \dots, \nu_{N-2}$:

$$i(N-i)\nu_{N-1}^2 = (N-1)\nu_i^2 + (N-i-1)\nu_i^2\nu_{N-1}^2, \quad i = 1, \dots, N-2,$$

the weight matrix W has another symmetric second order differential operator

$$D_2 = \partial^2 G_2(t) + \partial^1 G_1(t) + \partial^0 G_0,$$

where the coefficients are given by

$$\begin{aligned} G_2(t) &= t(J - At), \\ G_1(t) &= ((1+\alpha)I + J)J + Y - t(J + (\alpha+2)A + Y^* - AY + YA), \\ G_0 &= \frac{N-1}{\nu_{N-1}^2}[J - (\alpha I + J)A], \end{aligned}$$

A and J are defined in (3.13), and $Y = \sum_{i=1}^{N-1} \frac{i(N-i)}{\nu_i} E_{i+1,i}$. Note that now the only free parameter is ν_{N-1}.

Let us define the Hermitian positive semidefinite matrix (and singular) $M = v^*v$, where v is a row vector

$$v = \sum_{j=1}^{N-1} \Big(\prod_{k=1}^{N-j} \frac{\nu_{N-k}(\alpha+k)}{k} \Big) e_j + e_N,$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the j-th unit vector. A simple computation gives

$$M = \sum_{i,j=1}^{N} \Big(\prod_{k=\min\{i,j\}}^{\max\{i,j\}-1} \frac{\nu_k(\alpha+N-k)}{N-k} \Big) \Big(\prod_{k=1}^{N-\max\{i,j\}} \frac{\nu_{N-k}(\alpha+k)}{k} \Big)^2 E_{ij}$$

(where for $m > n$ we take $\prod_{k=m}^{n} = 1$).

Let D be the following differential operator $D = -(N-1)D_1 + D_2$. As a linear combination of symmetric operators with respect to the weight matrix (3.12), D is also symmetric with respect to this weight matrix. The differential coefficients $F_i(t)$, $i = 0, 1, 2$, of D are

$$\begin{aligned} F_2(t) &= t(-(N-1)I + J - At), \\ F_1(t) &= Y - \sum_{i=1}^{N} (i-1)(\alpha+N-i+1)E_{ii} + t(J - (\alpha+N+1)A - Y^*), \\ F_0 &= \frac{(N-1)(1+\nu_{N-1}^2)}{\nu_{N-1}^2}[J - (\alpha I + J)A]. \end{aligned}$$

Evaluating them at $t = 0$ and considering the bidiagonal structure of $F_1(0)$ and F_0, it is easy to check that $F_1(0)v^* = 0$ and $F_0 v^* = 0$. Thus the conditions (2.6) are satisfied ($F_2(0) = 0$).

Hence Theorem 2.2 implies that D is symmetric with respect to any of the following weight matrices:

$$W_{\gamma,\zeta}(t) = \gamma W(t) + \zeta\delta(t)M, \quad t \in (0, +\infty), \quad \alpha > -1, \quad \gamma > 0, \zeta \geq 0.$$

We illustrate the case of size 2×2 ($\nu_1 = a$). The weight matrix $W_{\gamma,\zeta}$ is

$$W_{\gamma,\zeta}(t) = \gamma t^\alpha e^{-t} \begin{pmatrix} t(1+a^2t) & at \\ at & 1 \end{pmatrix} + \zeta \begin{pmatrix} a^2(\alpha+1)^2 & a(\alpha+1) \\ a(\alpha+1) & 1 \end{pmatrix} \delta(t),$$

while the second order differential operator D is

$$D = \partial^2 \begin{pmatrix} 0 & -at^2 \\ 0 & -t \end{pmatrix} + \partial^1 \begin{pmatrix} t & -\dfrac{(1+a^2(\alpha+3))t}{a} \\ \dfrac{1}{a} & -(\alpha+1) \end{pmatrix} + \partial^0 \begin{pmatrix} \dfrac{a^2+1}{a^2} & -\dfrac{(1+a^2)(\alpha+1)}{a} \\ 0 & 0 \end{pmatrix}.$$

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