A singular element for three-dimensional fracture mechanics analysis

M. P. Ariza, A. Sáez & J. Domínguez*
Escuela Superior de Ingenieros, Universidad de Sevilla, Av. Reina Mercedes s.n. 41012-Sevilla Spain

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In this article, a singular boundary element for three-dimensional fracture mechanics analysis is presented. It is a nine-node quadratic element with plane geometry. These nodes are located at one quarter of the distance between two opposite sides of the element. Shape functions with a $1/r$ singularity at the crack front are used to represent the tractions. The Stress Intensity Factors are computed as system unknowns appearing (except for a constant) as traction nodal values. Special attention is paid to the development of a simple and accurate integration approach for this singular element. The accuracy of the results obtained with the proposed element is demonstrated by solving several crack problems including edge and embedded cracks with different geometries. The element can be easily implemented and incorporated into existing quadratic boundary element codes. In a companion paper the element is formulated and used for fracture mechanics problems in transversely isotropic materials. Extension to other fields for which boundary element formulations exist, is quite simple. © 1998 Elsevier Science Ltd. All rights reserved

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1 INTRODUCTION

Fracture mechanics has been an important research field in engineering during the last fifty years. The interest was induced first by the appearance of catastrophic failures caused by crack propagation and more recently by the need for internal damage evaluation in engineering materials in order to guarantee the safety of existing structures and machines for the longest possible period of time. Stress Intensity Factor (SIF) computation is of fundamental concern since this parameter is the best and most extended indicator of crack propagation.

The number of fracture problems with a close form analytic solution is very small. Because of that, different numerical methods suitable for this kind of problem were presented. First the Finite Element Method (FEM) and more recently the Boundary Element Method (BEM) have become well known and very useful tools for the study of fracture mechanics problems. Several techniques were developed within the FEM for SIF evaluation. In particular the formulation of quarter-point elements by Barsoum and Shih et al.2 for two-dimensional problems, and by Ingraffea & Manu3 for three-dimensions, was of great importance. At the same time the BEM appeared as a promising alternative (Cruse4,5). Blandford et al.6 developed a quarter-point boundary element for two-dimensional SIF evaluation from the quarter-point idea of the FEM. They used the standard boundary element formulation in combination with a subdomain technique to avoid the difficulties produced by a boundary (the crack surface) with two coincident parts. Martinez & Domínguez7 presented a new way of computing the SIF directly as a boundary element system of equations unknown. Several studies using the hypersingular formulation of the BEM for SIF computation, were also presented during the last few years. Combining the standard and the hypersingular formulations of the BEM, only the external and the crack boundaries have to be discretized. Thus, the need of sub-regioning is avoided. Portela et al.8 presented the effective implementation of this Dual Boundary Element approach for two-dimensional crack problems and used discontinuous elements and the J-integral to compute values of the SIF. Sáez et al.9 presented more general curved quadratic as well as quarter-point elements with the same hypersingular formulation and computed the SIF from the

*Corresponding author.
Crack Opening Displacements (COD) at collocation points extremely close to the tip.

All the earlier mentioned BEM approaches are formulated for two-dimensional problems. The BEM for three-dimensional crack problems has received much less attention up to now. Luchi & Rizzuti\textsuperscript{10} developed an eight-node quadratic element with special shape functions including singular traction variation in the vicinity of the crack front. Gaugming and Yougyuan\textsuperscript{11} presented a quarter-point boundary element formulation for circular and elliptical cracks. More recently Mi & Aliabadi\textsuperscript{12,13} have extended the use of the hypersingular formulation to three-dimensional problems. Using discontinuous elements and displacement extrapolation techniques these authors computed Stress Intensity Factors for several problems.

In the present article a nine-node singular quarter-point quadratic boundary element for three-dimensional crack problems is presented. The element is formulated for the standard displacement integral representation and the sub-domain technique is used when the problem is non-symmetric. It is the three-dimensional counterpart of the quarter-point element introduced by Blandford \textit{et al.}\textsuperscript{8} and Martínez & Domínguez\textsuperscript{7}. The crack displacement variation is directly represented by locating the mid-nodes at one quarter of the element width. The tractions over the elements next to the crack front are represented by singular functions obtained by dividing the quadratic shape functions by the square root of the distance to the front. A simple integration approach is developed. It allows for a straightforward and accurate integration of the singular kernels over the quarter-point elements with singular traction shape functions. Values of the SIF are computed as system unknowns at the crack front nodes. They can also be computed from the Crack Opening Displacements at nodes inside the crack.

As compared to the element presented by Luchi & Rizzuti\textsuperscript{10}, which is also based on the displacement formulation of the BEM and produces accurate results, the present formulation is, to the authors' judgement, simpler and easier to implement in a standard BE code. In the element presented by those authors there are eight nodes, the side nodes are placed at the middle of the side, and special shape functions are defined to represent displacements and tractions. The use of quarter-point elements and singular quarter-point elements, as done in the present article, leads to an integration process with little variation with respect to the general procedure, it makes unnecessary any change of special transformation when integration is done from outside quarter-point or singular quarter-point elements or when it is done from inside a quarter-point element. Only when the integration is done from inside a singular quarter-point element, a simple transformation process is required to cancel all the singularities. These features make the present approach simple and very easy to include into existing general quadratic three-dimensional Boundary Element codes.

Several numerical examples are studied using the present approach. The computed results are very accurate as compared to those obtained by other authors using different techniques. The main advantages of the element presented in this article are: a simple formulation and implementation, a straightforward connection with other quadratic elements, and a simple and accurate Stress Intensity Factor computation in any three-dimensional crack problem.

2 BASIC EQUATIONS

The displacement integral representation for a point 'i' inside or on the boundary $\Gamma$ of an elastic body $\Omega$, can be written as:

$$c_i^j u_i^j + \int_{\Gamma} \rho_i^j u_i d\Gamma = \int_{\Gamma} u_i^j p_i d\Gamma$$

where $u_i$ and $p_i$ are the components of the displacement and traction vectors, respectively; $u_i^j$ and $p_i^j$ are the functional solution displacement and traction tensors, respectively, and $c_{ij}$ is the local tensor at 'i' such that $c_{ii} = (1/2)\delta_{ii}$ for smooth boundary points and $c_{ii} = \delta_{ii}$ for internal points. Expressions for the fundamental solution tensors can be found, for instance, in the book by Brebbia and Dominguez\textsuperscript{14}. Using vector notation, the earlier equation can be written as:

$$c^j u^j + \int_{\Gamma} p^j u d\Gamma = \int_{\Gamma} u^j p d\Gamma$$

where $u$ and $p$ are vectors containing the three displacement and traction components, respectively; and $c$, $u^j$ and $p^j$ are 3 x 3 tensors.

3 BOUNDARY ELEMENTS

The boundary $\Gamma$ of the body $\Omega$ is divided into NE elements. An element 'j' contains NJ nodes. Displacements $u$ and tractions $p$ over each boundary element are written in terms of their nodal values and shape functions. Thus,

$$u = \Phi u^j \quad \text{and} \quad p = \Phi p^j$$

(3)

where $u^j$ and $p^j$ are vectors containing the 3NJ nodal values of $u$ and $p$, respectively, and $\Phi$ is a $(3 \times 3NJ)$ matrix containing the shape functions over the element 'j'. In the case under consideration, the elements contain six nodes when they are triangular or nine nodes when they are quadrilateral (Fig. 1). In the latter case, the shape functions in terms of the natural coordinates $-1 \leq \xi_1 \leq 1, -1 \leq \xi_2 \leq 1, \xi_3 \leq 1$, are:

$$\phi_1 = \frac{1}{2} \xi_1 (\xi_1 - 1) \xi_2 (\xi_2 - 1); \phi_2 = \frac{1}{2} (1 - \xi_1 ) \xi_2 (\xi_2 - 1);$$

$$\phi_3 = \frac{1}{4} \xi_1 (\xi_1 + 1) \xi_2 (\xi_2 - 1);$$

$$\phi_4 = \frac{1}{2} \xi_1 (\xi_1 + 1)(1 - \xi_2^2); \phi_5 = \frac{1}{4} \xi_1 (\xi_1 + 1) \xi_2 (\xi_2 + 1);$$

$$\phi_6 = \frac{1}{2} (1 - \xi_1^2 ) \xi_2 (\xi_2 + 1);$$

The crack displacement variation is represented by the square root of the distance to the front. A simple transformation process is required to cancel all the singularities. These features make the present approach simple and very easy to include into existing general quadratic three-dimensional Boundary Element codes.
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After discretization, eqn (2) becomes:

\[
\sum_{j=1}^{N} \left\{ \int_{\Gamma_j} \mathbf{u}^* \Phi \mathrm{d}\Gamma \right\} \mathbf{u}^t = \sum_{j=1}^{N} \left\{ \int_{\Gamma_j} \mathbf{p}^* \Phi \mathrm{d}\Gamma \right\} \mathbf{p}^t
\]

which can be written for all the boundary nodes to yield the system of equations

\[
\mathbf{H} \mathbf{U} = \mathbf{G} \mathbf{P}
\]

where \( \mathbf{U} \) and \( \mathbf{P} \) contain all the nodal values of the problem, and \( \mathbf{H} \) and \( \mathbf{G} \) the integrals of the fundamental solution tensors over the boundary elements. The element geometry is also quadratic with isoparametric representation:

\[
\mathbf{x} = \Phi \mathbf{x}'
\]

where \( \mathbf{x}' \) contains the 3NJ cartesian coordinates of the nodes in element 'j'.

Matrices \( \mathbf{G} \) and \( \mathbf{H} \) consist of integrals over each element 'j':

\[
\mathbf{G} = \int_{\Gamma_j} \mathbf{u}^* \Phi \mathrm{d}\Gamma = \int_{\xi_1} \int_{\xi_2} \mathbf{u}^* \Phi \mathbf{G} \mathrm{d}\xi_1 \mathrm{d}\xi_2
\]

\[
\mathbf{H} = \int_{\Gamma_j} \mathbf{p}^* \Phi \mathrm{d}\Gamma = \int_{\xi_1} \int_{\xi_2} \mathbf{p}^* \Phi \mathbf{G} \mathrm{d}\xi_1 \mathrm{d}\xi_2
\]

where \( \mathbf{G} \) is the Jacobian of the transformation of the element geometry from cartesian coordinates into natural coordinates \( \xi_1, \xi_2 \) (Fig. 1) in accordance with the geometry definition (7). These integrals are easily computed for collocation points which are not inside the element 'j' by using a Gauss numerical quadrature. When the collocation point \( 'i' \) belongs to the integration element 'j', the kernels \( \mathbf{u}^* \) and \( \mathbf{p}^* \) contain singularities of the type \( \frac{1}{r} \) and \( \frac{1}{r^2} \), respectively, \( r \) being the distance to the collocation point. In this case, the integration domain is subdivided into triangles and subsequent transformation of these triangles into squares yields a Jacobian of the type \( O(r) \) which cancels out the singularity (Fig. 2) and allows the use of standard Gauss integration to evaluate \( \mathbf{G} \). The computation of the integrals \( \mathbf{H} \) containing a singularity is avoided by using the zero traction condition under rigid body motion. Details of this integration process can be found in Domínguez.15

4 SINGULAR QUARTER-POINT ELEMENT

According to Linear Elastic Fracture Mechanics, displacements and stresses near the front of a crack have a \( \sqrt{r} \) and \( \frac{1}{\sqrt{r}} \) variation, respectively, when the distance to the front \( r \) tends to zero. To have an adequate representation of these displacements and stresses, a nine-node quadratic boundary element was developed. This type of element is used on the first row at both sides of the crack front (note that an internal boundary is always introduced by cutting the domain through the crack). The element (Fig. 3) is quadrilateral and has its nine nodes on a plane. Each one of the nodes 4, 8 and 9 is located on a straight line and at a distance of one quarter between 3 and 5, 1 and 7, and 2 and 6, respectively. Sides 1, 8, 7 and 3, 4, 5 are perpendicular to the line 1, 2, 3 which follows the crack front. The distance between 1 and 7 is the same as between 3 and 5. When these conditions are satisfied, the following relation between the variable \( \xi_2 \) and the distance \( r \) to the front 1, 2, 3 can be easily obtained from (7):

\[
\xi_2 = 2(\sqrt{rL}) - 1
\]

where \( L \) is the distance between nodes 1 and 7.

Displacements, represented by eqns (3) and (4), become in this case:

\[
u_i = a_i^1 + a_i^2 \sqrt{rL} + a_i^3 \frac{p}{rL}
\]

where \( a_i^m \) are polynomial functions of the nodal values of \( u_i \) and the element coordinate \( \xi_1 \) along the crack front.

In order to obtain a traction variation over the element which is able to represent accurately the tractions near the crack front, the shape functions (4) are modified to include the \( 1/\sqrt{rL} \) singularity. To do so, each shape function is divided by \( (\xi_2 + 1)/2 \); i.e., by \( \sqrt{rL} \) and the tractions representation written as:

\[
p = \Phi \mathbf{p}'
\]

where \( \Phi \) contains the modified shape functions and \( \mathbf{p}' \) the nodal values which are the values of \( p \) at the nodes of element 'j' multiplied by \( (\xi_2 + 1)/2 = \sqrt{rL} \) at each node.
Thus, the values contained in the modified $p'$ for nodes 1, 2 and 3 represent the SIF, except for a constant.

The variation of each traction component $p$, with the distance to the crack from $r$ which can be obtained from (11) is

$$p_i = \tilde{a}_i^1 (1/\sqrt{rL}) + \tilde{a}_i^2 + \tilde{a}_i^3 (\sqrt{rL})$$

(12)

where $\tilde{a}_i^n$ are polynomial functions of the nodal values $\tilde{p}$, of the tractions representation and the element $\xi_i$. For instance, along the line between nodes 1 and 7 ($\xi_1 = -1$) eqn (12) gives the variation of $p$, with the distance to the crack front $r_i$ with $\tilde{a}_1^1 = \tilde{p}_1^3$, $\tilde{a}_1^2 = \tilde{p}_1^3 + 4\tilde{p}_1^6 - 3\tilde{p}_1^2$ and $\tilde{a}_1^3 = 2\tilde{p}_1^6 - 4\tilde{p}_1^2$.

Therefore, $\tilde{p}_1^1$ gives (except for a constant) the Stress Intensity Factor at node 1. Similarly, along the line 2 – 9 – 6 ($\xi_1 = 0$), $\tilde{a}_2^1 = \tilde{p}_2^3$, $\tilde{a}_2^2 = -\tilde{p}_2^3 + 4\tilde{p}_2^6 - 3\tilde{p}_2^2$ and $\tilde{a}_2^3 = 2\tilde{p}_2^6 - 4\tilde{p}_2^2$, and $\tilde{p}_2^1$ gives the SIF values at node 2. The same can be said for $\tilde{p}_1^1$ at node 3.

5 BOUNDARY ELEMENT DISCRETIZATION AND SIF COMPUTATION

The Boundary Element representation of three-dimensional crack problems is done in the following way: (i) the domain under consideration is divided into sub-domains by cutting it by a section through the crack; (ii) each sub-domain is analysed by the BEM; and (iii) the sub-domains are coupled using equilibrium and compatibility relations. When the crack is in a plane of symmetry (Fig. 4), these relations become simple-symmetry conditions applied over one sub-region.

The elements used are standard quadratic elements except for those with one side at the crack front. Among these elements, those inside the crack (zero traction) are quarter-point elements. Thus, the displacements are accurately represented by eqn (10). The tractions on these elements do not require any particular treatment; they are represented using the general procedure. The elements with one side along the crack front which are part of the boundary inside the material, are singular quarter-point elements. This type of element is the only one which requires a special treatment of the fundamental solution integrals.

Once the boundary value problem was solved and the displacement and traction nodal values are known, the SIF

![Fig. 2. Element subdivision for local integration over quadratic elements.](image)

![Fig. 3. Nine node quarter-point quadratic element.](image)
can be computed directly from the traction nodal values at the crack front. Assuming that the $x_3$ axis is perpendicular to the crack and $x_2$ is tangent to the crack front line at the point where a node $k$ is located, the Stress Intensity Factors at this point are:

$$K_i = \bar{\sigma}_i \sqrt{2\pi L}$$
$$K_{ii} = \bar{\sigma}_i \sqrt{2\pi L}$$
$$K_{iii} = \bar{\sigma}_i \sqrt{2\pi L}$$

(13)

where $\bar{\sigma}_i$ is the 'i' component of the traction nodal value at node $k$, and $L$ is the length of the singular quarter-point element in the direction perpendicular to the crack front.

The SIF can also be computed from the displacements inside the crack. Taking into account that the boundary element displacement representation (10) reproduces the exact displacement variation near the crack front, one only has to particularize eqn (10) for a nodal line perpendicular to the crack front, or for a quarter-point node, and make it equal to the analytical solution in terms of the SIF for a line perpendicular to the crack front, or for the quarter-point node. From this equation one obtains the SIF in terms of the displacement nodal values at the two nodes along the line perpendicular to the front, in the first case, or at the quarter-point node, in the second. In this paper the second approach was used. Thus

$$K_i = \frac{\mu}{1-\nu} \sqrt{\frac{2\pi}{L}} u_i$$
$$K_{ii} = \frac{\mu}{1-\nu} \sqrt{\frac{2\pi}{L}} u_i$$
$$K_{iii} = \frac{\mu}{1-\nu} \sqrt{\frac{2\pi}{L}} u_i$$

(14)

where $\mu$ is the material shear modulus, $\nu$ the Poisson's modulus and $u_i$ the displacement component 'i' at the quarter-point node 'q' when the crack is in a plane of symmetry. Otherwise, $u_i$ represents one half of the crack opening displacement in the 'i' direction at the quarter-point node 'q'.

6 INTEGRATION OVER THE SINGULAR QUARTER-POINT ELEMENTS

The integration over non-singular elements, whether or not they are quarter-point elements, is always done in a simple and general way by using a Gauss quadrature. The integration over singular quarter-point elements does not present particular difficulties when the collocation point is outside the integration element. The $1/\sqrt{r}$ singularity of the shape functions is offset by the Jacobian of the transformation from cartesian to $\xi_1, \xi_2$ coordinates. This fact is a consequence of the location of the mid nodes at one quarter of the length of the elements in the direction perpendicular to the crack front.

The integration over the singular quarter-point element requires special treatment when the collocation point is one of the nodes of the element. Several simple transformations of the integration domain are required to cancel all the singularities prior to the application of Gauss quadrature (Fig. 5).

Transformation 1:

The integration element is transformed into a square domain in terms of the natural coordinates $-1 \leq \xi_1 \leq 1$ and $-1 \leq \xi_2 \leq 1$. To do so, a quadratic transformation in the direction of the crack front and linear in the perpendicular direction is defined. This is possible for singular quarter-point elements because the nodes follow a straight line in the latter direction. The relation between cartesian and natural coordinates is defined by eqn (7) but
the shape functions are now:

\[ \phi_1 = \frac{1}{4} \eta_1 (\eta_1 - 1)(1 - \eta_2); \phi_2 = \frac{1}{2}(1 - \eta_1^2)(1 - \eta_2); \]

\[ \phi_3 = \frac{1}{4} \eta_1 (\eta_1 + 1)(1 - \eta_2); \phi_4 = 0; \phi_5 = \frac{1}{2}(1 - \eta_1^2)(1 + \eta_2); \]

\[ \phi_7 = \frac{1}{4} \eta_1 (\eta_1 - 1)(1 + \eta_2); \phi_8 = 0; \phi_9 = 0 \]  \hspace{1cm} (15)

**Transformation 2:**

Once the integration domain is a square, it is subdivided into triangles with a vertex at the collocation node. Fig. 5 shows the case when the collocation point is node 2. Next, each triangle is transformed into a square with the collocation point transforming into one side of the square. This transformation can be written as:

\[ \eta = (1 - s_1) \eta_1 + s_1 (1 - \eta_1) \eta_2 + s_2 \eta_3 \]  \hspace{1cm} (16)

where \( 0 \leq s_1 = 1, 0 \leq s_2 = 1 \) and, \( \eta_1, \eta_2 \) and \( \eta_3 \) are the \( \eta \) coordinates of the collocation point 1, and the other two vertex of the triangle (Fig. 5). The collocation point 1 becomes the side \( s_1 = 0 \). The Jacobian of this transformation is:

\[ \left| J \right| = \frac{dy_1 dy_2}{ds_1 ds_2} = 2A s_1 \]  \hspace{1cm} (17)

A being the area of the triangle and \( s_1 \) the coordinate, in the transformed domain, proportional to the distance \( r \) to the collocation point in the triangle. Thus, the Jacobian of this transformation compensates the \( 1/r \) singularity of the fundamental solution tensor \( u_{ik} \).

**Transformation 3:**

After the first two transformations there is still a singular part in the kernels of the integrals. The term \( 1/\sqrt{r} \), \( \tilde{r} \)

being the distance to the crack front, has not been counteracted. According to the first transformation, which is linear in the direction perpendicular to the crack front (eqns 7 and 15),

\[ \tilde{r} = (1 + \eta_2) L/2 \]  \hspace{1cm} (18)

The relation between \( \eta_2 \) and the two coordinates \( s_1, s_2 \) introduced in the second transformation is defined by eqn (16). Depending on the collocation node and the triangle in which the integration is carried out, the expressions for \( \eta_2 \) in terms of \( s_1, s_2 \) are different and therefore the expressions for \( \tilde{r} \) are also different. These expressions for \( r \) in terms of \( s_1, s_2 \) are given in the Appendix for all possible situations. In all cases, changes of variables of the type:

\[ s_1 = s_1^2 \text{ or } (1 - s_1) = (1 - s_1)^2 \]

and/or

\[ s_2 = s_2^2 \text{ or } (1 - s_2) = (1 - s_2)^2 \]

produce a Jacobian

\[ \left| J \right| = \frac{dy_1 dy_2}{ds_1 ds_2} \]  \hspace{1cm} (20)

which counteracts the \( 1/\sqrt{r} \) variation when \( \tilde{r} \rightarrow 0 \). The change of variables and the Jacobian for each situation are given in the Appendix.

**Transformation 4:**

To obtain the standard Gauss quadrature integration domain, simple linear changes are defined

\[ s_1^* = \frac{t_1^* + 1}{2} \text{ and } s_2^* = \frac{t_2^* + 1}{2} \]  \hspace{1cm} (21)

which completes the necessary transformation process.

By means of transformations 1 to 4 a kernel without singularities is obtained and the integration of \( u_{ik} \) over the singular quarter-point element when the collocation node belongs to the integration element, can be evaluated numerically. The integrals of \( t_{ik}^* \) over the singular
quarter-point element, to obtain the corresponding HW matrices, are evaluated indirectly by using the zero traction condition under rigid body motion.

7 NUMERICAL EXAMPLES

Several three-dimensional crack problems have been analysed using the approach presented in this paper. In all the cases the boundary is discretized using standard quadratic quadrilateral or triangular elements except for the two rows of elements in contact with the crack front. The elements inside the crack and having one side on the crack front, are quarter-point elements. Those having one side on the crack front but being inside the solid as part of the internal boundary introduced for sub-regioning, are singular quarter-point elements.

Example 1: Cylinder with symmetrical penny shape internal crack under uniform traction

Fig. 4 shows the geometry of the problem and the boundary element discretization of one eighth of the domain. Symmetry boundary conditions are applied to the discretized subregion. The crack surface is discretized using four rows of elements with the same width. The elements in the forth row are quarter-point elements. The elements in the fifth row (first row outside the crack) are singular quarter-point. The material Poisson’s ratio is $\nu = 0.3$. The SIF computed from the displacement at the quarter-point node (eqn 14) is $K_1 = 0.698 \sigma \sqrt{\pi a}$ and the one obtained from the traction nodal value at the crack front is $K_1 = 0.7064 \sigma \sqrt{\pi a}$, where $\sigma$ is the applied traction and $a$ the crack radius. Both results, and in particular the second one, are in very good agreement with the $K_1 = 0.706 \sigma \sqrt{\pi a}$ value obtained by Banks-Sills and Sherman using the stiffness derivative and the J-integral methods.

Example 2: Prismatic plate with embedded elliptical crack under uniform traction

The geometry of this problem is shown in Fig. 6. The dimensions of the plate and the crack are $b/a = 0.4$, $b/t = 0.2$; $a/w = 0.1$ and $h/w = 1$. The material Poisson’s ratio is $\nu = 0.3$ and the load is a uniform traction applied on the two opposite sides parallel to the plane of the crack. This problem was studied by Newman and Raju using several Finite Element discretizations. These authors computed SIF values by the nodal force method. The boundary element mesh employed for one eighth of the body is shown in Fig. 7. Symmetry allows solution of the problem using this model. The mesh consists of 297 quadratic elements with 902 nodes. The eight elements with one side along the crack front and extending toward the interior of the crack are quarter-point elements and the eight elements with one side along the crack front and extending inside the body are singular quarter-point elements. To verify the accuracy of the solutions, convergence was studied by varying the size of the quarter-point and the singular quarter-point elements in the direction perpendicular to the crack front. Results for element size from $L/b = 0.0625$ to $L/b = 0.375$ were obtained. Values of $K_l(q^2/Q)^{0.5}$ computed with $L/b = 0.125$ and $L/b = 0.25$ are presented in Fig. 8 for different positions along the crack front. This position is denoted by the angle $\varphi$ measured with respect to the largest radius of the ellipse (Fig. 6). The SIF $K_l$ is computed from the traction nodal values (eqn 13) and $Q$ is the shape factor of the ellipse (which is the square of the complete integral of the second kind). The results shown in Fig. 8 correspond to nodes 1 and 3 of every singular quarter-point element along the crack front. These results are less sensitive to changes in the number of numerical integration points than values corresponding to nodes 2 of the same elements. The computed values for $K_l$ present very little dependence on the size of the elements in the direction perpendicular to the crack front, in the range studied. Results obtained for element size $L/b = 0.0625$ to $L/b = 0.375$ are very close to those shown in the figure for $L/b = 0.125$ and $L/b = 0.25$. All the computed values of $K_l$ are in agreement with those obtained by Newman and Raju which, according to these authors, are accurate within 3%.

Example 3: Prismatic plate with semi-elliptical surface crack under uniform traction

![Fig. 6. Prismatic plate with embedded elliptical crack.](image-url)
The geometry of this problem is shown in Fig. 9. It corresponds to one half of the previous example. The crack is now semi-elliptical and is on the surface of the plate. The relative dimensions of the crack and the plate are: \( b/a = 0.4, \) \( b/t = 0.2, \) \( a/w = 0.1 \) and \( h/w = 1. \) The material Poisson’s ratio is \( \nu = 0.3. \) The boundary element mesh used in this case is the same as in the previous example, shown in Fig. 7. This model corresponds now to one quarter of the plate. Values of \( K_I \) computed from the tractions at the singular quarter-point elements (eqn 13) are compared in Fig. 10 with those presented by Raju and Newman\(^{15} \) for the same problem.

**Example 4: Prismatic plate with rectangular edge crack under uniform traction**

Fig. 11 shows the geometry of this problem. The dimensions of the plate and the crack are: \( a = w = h = 2t. \) The material Poisson’s ratio is: \( \nu = 0.3. \) Due to the symmetry of the problem only one half of the plate is discretized using the 80 elements and 262 nodes mesh shown in Fig. 12. It contains two quarter-point elements and two singular quarter-point elements. The \( K_I \) SIF values are computed using eqn (13) at all the nodes along the crack front.

The two-dimensional solution for this problem \( (K_I = 3.01\sigma\sqrt{a}) \) is taken as a reference of comparison (Sáez et al.\(^{16} \)). Fig. 13 shows the \% difference between that solution and the one obtained in this paper at different positions along the crack front, when zero normal displacement and zero shear traction boundary conditions are assumed over the two lateral faces of the plate. A good agreement exists between both solutions.

**8 CONCLUSIONS**

In this paper, a singular boundary element for the analysis of three-dimensional fracture mechanics problems has been presented. It is a nine-node quadratic element with plane
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geometry. Three nodes are located at one quarter of the distance between the side of the element at the crack front and its opposite side. These two sides have quadratic geometry. Special shape functions with a $1/\sqrt{r}$ singularity at the crack front are used to represent the tractions whereas the $\sqrt{r}$ variation of the displacements at the crack surfaces is directly represented by the element. Using the proposed element, the Stress Intensity Factors have been evaluated in a simple and direct way as one of the unknowns of the system of equations. The SIFs are, except for a constant, the traction nodal values at the nodes on the crack front.

Special attention has been paid to the development of a simple numerical integration approach for this singular boundary element. Two singularities exist in the element when the collocation point is one of its nodes: one is due to the fundamental solution and the other to the singular shape functions used for the tractions representation. Both singularities are cancelled out by simple transformations of the integration domain. These transformations allow the subsequent integration by a simple Gauss numerical quadrature. The resulting integration approach is simple, accurate and reliable.

Solutions to several three-dimensional crack problems have been presented. The computed values of the SIF are very accurate for all the numerical examples. The obtained results show very good agreement with those obtained by other authors for elliptical, semi-elliptical, penny shape and rectangular cracks.

The main advantages of the present singular element are its robustness, its simplicity of formulation and the fact that it can be very easily incorporated into existing standard three-dimensional quadratic boundary element codes. The only transformation required is the inclusion of a routine for the integration over the singular element when the collocation point is one of its nodes. In a companion paper (Sáez et al. 19) the element is formulated for transversely isotropic materials and Stress Intensity Factors are obtained for crack problems in this kind of media.

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REFERENCES


**APPENDIX A** Change of coordinates for local integration in singular quarter-point elements (Fig. A).

| Collocation Node | Triangle | Change of coordinates | $|J|$ |
|------------------|----------|-----------------------|-----|
| 1                | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $2s_1^4$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2s_2^2$ |
| 2                | a        | $s_1 = s_1^2, 1 - s_2 = (1 - s_2)^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $2s_1^4$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2s_2^2$ |
| 3                | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $2s_1^4$ |
|                  | b        | $s_1 = s_1^2, 1 - s_2 = (1 - s_2)^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2s_2^2$ |
| 4                | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
| 5                | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
| 6                | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
| 7                | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
| 8                | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
| 9                | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
| 10               | a        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
|                  | b        | $s_1 = s_1^2, s_2 = s_2^2$ | $2(1 - s_1)$ |
|                  | c        | $s_1 = s_1^2, s_2 = s_2^2$ | $4s_1^2(1 - s_1^2)$ |
Fig. A. Element subdivision for numerical integration over singular quarter-point element. Load point from node 1 to node 9 in Figs. A.1 to A.9, respectively.