Integrable logarithmic connections with respect to a cusp

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First (motivating) lecture

Local Analytic Geometry

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Let $D \subset X = \mathbb{C}^2$ be the cusp defined by the equation $h = x^2 - y^3 = 0$. A basis of the logarithmic derivations with respect to $D$ is $\{\chi, \delta\}$ with $\chi = 3x\partial_x + 2y\partial_y$, $\delta = 3y^2\partial_x + 2x\partial_y$. We have $\chi(h) = 6h$, $\delta(h) = 0$ and $[\chi, \delta] = \delta$.

Let us denote by $\mathcal{V}_X$ the subring of $\mathcal{D}_X$ consisting of logarithmic differential operators with respect to $D$, i.e. $\mathcal{V}_X = \mathcal{O}_X[\delta_1, \delta_2]$.

Let us denote by $\mathcal{O} = \mathcal{O}_X, 0$, $\mathcal{D} = \mathcal{D}_X, 0$ and $\mathcal{V} = \mathcal{V}_X, 0$ the corresponding stalks at the origin.

Any $P \in \mathcal{V}$ can be expressed in a unique way as

$$P = \sum_{\alpha \in \mathbb{N}^2_{\text{finite}}} a_\alpha \chi^{\alpha_1} \delta^{\alpha_2}, \quad a_\alpha \in \mathcal{O}. \quad (1)$$

1 Integrable logarithmic connections of rank 1

An integrable logarithmic connection (with respect to $D$) of rank $d$ is given by a free $\mathcal{O}_X$-module $E$ of rank $d$, let us say with a basis $\{e_1, \ldots, e_d\}$, and an action of $\chi$ and $\delta$ given by matrices $A, B \in M_{d \times d}(\mathcal{O}_X)$:

$$\chi e^t = Ae^t, \quad \delta e^t = Be^t, \quad e = (e_1, \ldots, e_d)$$

satisfying the integrability condition:

$$\chi(B) = \delta(A) + [A, B] + B.$$ 

If $E$ is an integrable logarithmic connection of rank 1 with a basis $\{e\}$, the action of $\chi$ and $\delta$ is determined by $a, b \in \mathcal{O}$ with $\chi e = ae, \delta e = be$ and the integrability condition becomes

$$\chi(b) - \delta(a) = b.$$ 

(1.1) Lemma. For any germ at the origin $E$ of integrable logarithmic connection of rank 1, there is a basis $\{e\}$ of $E$ and a constant $\alpha \in \mathbb{C}$ such that $\chi e = \alpha e, \delta e = 0$.

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Proof. Let \( \{e'\} \) be a basis of \( \mathcal{E} \) and set \( \chi e' = a'e', \delta e' = b'e' \). Let \( u \in \Theta^X \) be a unit and set \( e = ue' \). We have
\[
\chi e = (\chi(u)u^{-1} + a')e, \quad \delta e = (\delta(u)u^{-1} + b')e.
\]

We are going to show that we can choose the unit \( u \) in such a way that \( \chi(u)u^{-1} + a' = a'(0,0) \). Let us write \( a' = \sum_w a'_w \) and \( u = \sum_w u_w \), where \( w \in 2\mathbb{N} + 3\mathbb{N} \) and the \( a'_w, u_w \) are quasi-homogeneous polynomials of weight \( w \) with respect to weights 3 and 2 for \( x \) and \( y \) respectively.

The above condition is equivalent to \( \chi(u) = u(a'_0 - a') \), i.e.
\[
w u_w = - \sum_{k=0}^{w-1} u_k a'_{w-k} \quad \forall w \in 2\mathbb{N} + 3\mathbb{N}.
\]

By choosing \( u_0 = 1 \) we obtain recursively all the \( u_w \). The convergence of the series can be proven in the following way. Let \( C, \rho \geq 1 \) be such that \( |a'_w| \leq C \rho^w \) for all the weights \( w \), where \( \| - \| \) is the supremum norm on a small poly-disc centered at the origin. Assume that \( |u_k| \leq (C \rho)^k \) for all \( k < w \). Then,
\[
|u_w| \leq \frac{1}{w} \sum_{k=0}^{w-1} |u_k| |a'_{w-k}| \leq \frac{1}{w} \sum_{k=0}^{w-1} (C \rho)^k C \rho^{w-k} = \frac{1}{w} \left( \sum_{k=0}^{w-1} C^{k+1} \right) \rho^w \leq (C \rho)^w.
\]

Let us take \( \alpha = a'_0 \) and \( e = ue' \). By construction we have \( \chi e = \alpha e \). Let \( b \in \Theta \) be such that \( \delta e = be \). By the integrability condition we have \( \chi(b) - b = \delta(\alpha) = 0 \), but \( \chi - 1 \) is injective and so \( b = 0 \).

Q.E.D.

(1.2) Definition. Let \( \alpha \in \mathbb{C} \) be any constant. We denote by \( \mathcal{E}_\alpha = \Theta_X h^\alpha \) the integrable logarithmic connection (with respect to \( D \)) of rank 1 whose underlying \( \Theta_X \)-module is the free \( \Theta_X \)-module with basis \( h^\alpha \) and the action of logarithmic derivations is given by
\[
\chi h^\alpha = 6 \alpha h^\alpha, \quad \delta h^\alpha = 0.
\]

The germ at the origin of \( \Theta_X h^\alpha \) will be denoted by \( \Theta h^\alpha \).

It is clear that \( \mathcal{E}_\alpha \) is generated as \( \mathcal{V}_X \)-module by \( h^\alpha \).

(1.3) Exercise. Prove that \( \text{ann}_{\mathcal{V}_X} h^\alpha = \mathcal{V}_X (\chi - 6\alpha, \delta) \) and so \( \mathcal{E}_\alpha \simeq \mathcal{V}_X / \mathcal{V}_X (\chi - 6\alpha, \delta) \).

Proof. Hint: we proceed locally at any point \( p \in D \). If \( p = (x_0, y_0) \neq (0,0) \), by the inverse function theorem, we can take a new system of local coordinates centered at \( p \).
\[
x = x - x_0, \quad y = h(x, y).
\]

Then, the expression of \( \chi \) and \( \delta \) in terms of \( (x, y) \) is
\[
\chi = 3(x + x_0) \partial_x + 6y \partial_y, \quad \delta = 3y^2 \partial_x.
\]

Consequently, \( \mathcal{V}_{X,p} (\chi - 6\alpha, \delta) = \mathcal{V}_{X,p} (y \partial_y - \alpha, \partial_x) \), but the expression of \( h \) in the new local coordinates is \( h = y \), and so \( \mathcal{V}_{X,p} = \Theta_{X,p} [\partial_x, y \partial_y] \). One easily sees that \( \text{ann}_{\mathcal{V}_{X,p}} y^\alpha = \mathcal{V}_{X,p} (y \partial_y - \alpha, \partial_x) \).

If \( p = (0,0) \) one has to use the expression \( \boxplus \) and a division argument by \( \chi - 6\alpha, \delta \).

Q.E.D.

(1.4) Proposition. With the above notations, there are natural isomorphisms of left \( \mathcal{V} \)-modules:
1. \( \mathcal{E}_\alpha \otimes_{\mathcal{E}_X} \mathcal{E}_\beta \simeq \mathcal{E}_{\alpha + \beta}, \alpha, \beta \in \mathbb{C} \).

2. \( \mathcal{E}_{-k} \simeq \mathcal{O}_X(kD), k \in \mathbb{Z} \).

3. \( (\mathcal{E}_\alpha)^* = \text{Hom}_{\mathcal{O}_X}(\mathcal{E}_\alpha, \mathcal{O}_X) \simeq \mathcal{E}_{-\alpha} \).

(1.5) **Exercise.** Prove the preceding proposition. One has to remember the definition of the internal \( \text{Hom}_{\mathcal{O}_X}(-, -) \) and tensor product \(- \otimes_{\mathcal{O}_X} -\) of integrable logarithmic connections.

(1.6) **Definition.** We can also consider the left \( \mathcal{T}_X[s] \)-module \( \mathcal{O}_X[s]h^s \), where the action of logarithmic derivations is the obvious one. It is a sub-\( \mathcal{T}_X[s] \)-module of the Bernstein module \( \mathcal{O}_X[h^{-1}, s]h^s \). It is clear that the annhilator of \( h^s \) over \( \mathcal{T}_X[s] \) is the left ideal generated by \( \chi - 6s \) and \( \delta \), and so \( \mathcal{O}_X[s]h^s \simeq \mathcal{T}_X[s]/\mathcal{T}_X[s](\chi - 6s, \delta) \) (prove it!).

(1.7) **Proposition.** The annihilator of \( h^s \) over \( \mathcal{D}_X[s] \) is the left ideal generated by \( \chi - 6s \) and \( \delta \), or in other words, the canonical map

\[
Q \otimes (ah^s) \in \mathcal{D}_X[s] \otimes_{\mathcal{T}_X[s]} (\mathcal{O}_X[s]h^s) \mapsto Q(ah^s) \in \mathcal{D}_X[s] \cdot h^s \subset \mathcal{O}_X[h^{-1}, s]h^s
\]

is an isomorphism.

(1.8) **Exercise.** Prove the preceding proposition. One should understand the expression we obtain when we evaluate a differential operator on the symbol \( h^s \).

One also has to use the fact that in this case \( h_x, h_y \) form a regular sequence.

The Bernstein polynomial (global = local at the origin) of \( h \) is \( b(s) = (s + 1)(s + 5/6)(s + 7/6) \) and an explicit functional equation is given by

\[
b(s)h^s = P(h^{s+1}), \quad P = \frac{1}{2}x\partial_x^2 + \frac{1}{2}y\partial_x^2\partial_y - \frac{1}{4\pi} \partial_x^2 + \frac{3}{4} \partial_y^2.
\]

Let us call \( \mathcal{M}_\alpha = \mathcal{D}_X \otimes_{\mathcal{T}_X} \mathcal{E}_\alpha = \mathcal{D}_X/\mathcal{D}_X(\chi - 6\alpha, \delta) \). Let us also call \( \iota_\alpha : \mathcal{E}_\alpha \hookrightarrow \mathcal{E}_{-1} \) the map given by \( ah^\alpha \mapsto (ah)h^{\alpha-1} \), which induces a natural map \( \mathcal{M}_\alpha \to \mathcal{M}_{\alpha-1} \) sending \( Q \) to \( Qh \).

(1.9) **Proposition.** Let \( \alpha \in \mathbb{C} \) be a constant such that \( b(\alpha - i) \neq 0 \) for all integers \( i \geq 1 \). Then the natural maps

\[
g_\alpha : Q \otimes (ah^\alpha) \in \mathcal{M}_\alpha \mapsto Q(ah^\alpha) \in \mathcal{O}_X[h^{-1}]h^\alpha = \mathcal{E}_\alpha[\star D]
\]

\[
\mathcal{M}_\alpha \to \mathcal{M}_{\alpha-1} \to \cdots \to \mathcal{M}_{\alpha-k} \to \cdots, \quad k \in \mathbb{N}
\]

are isomorphisms. In particular, \( \mathcal{E}_\alpha[\star D] = \mathcal{O}_X[h^{-1}]h^\alpha = \mathcal{D}_Xh^{\alpha-k} \) for \( k \in \mathbb{N} \).

**Proof.** For the surjectivity of \( g_\alpha \) it is enough to show that for any integer \( k > 0 \) there is a differential operator \( L \) such that \( L(h^\alpha) = h^{-k}h^\alpha \), but from the Bernstein functional equation (2) we can take

\[
L = b_k(\alpha)^{-1}P_k, \quad b_k(s) = b(s - k) \cdots b(s - 2)b(s - 1).
\]

Let \( Q \in \mathcal{D} \) be such that \( Q(h^\alpha) = 0 \). The action of \( Q \) on \( h^s \) gives \( Q(h^s) = q(s)h^{-d}h^s \), with \( q(s) \in \mathcal{O}[s], \ d = \text{ord}Q \). Since \( Q(h^\alpha) = 0 \) we have \( q(\alpha) = 0 \).
and so $q(s) = (s - \alpha)q'(s)$, $q'(s) \in \mathcal{O}[s]$. From the functional equation again we obtain
\[ b_d(s)h^{-d}h^s = P^d(h^s) \]
and $b_d(s)Q - (s - \alpha)q'(s)P^d \in \text{ann}_{\mathcal{O}[s]} h^s = \mathcal{O}[s](\chi - 6s, \delta)$. By taking $s = \alpha$ we deduce $b_d(\alpha)Q \in \mathcal{O}(\chi - 6\alpha, \delta)$, but from our hypotheses $b_d(\alpha) \neq 0$ and so $Q \in \mathcal{O}(\chi - 6\alpha, \delta)$. This shows that the map $\varrho_\alpha$ is injective.

With the same argument we obtain that $\varrho_{\alpha - i}$ is an isomorphism for all integers $i \geq 1$, and so the maps $\mathcal{M}_{\alpha - i+1} \rightarrow \mathcal{M}_{\alpha - i}$ are also isomorphisms. Q.E.D.

(1.10) Question. The above proposition is true for any germ of hypersurface $h = 0$ in any dimension. Is it true the opposite? i.e. if the natural maps
\[ \mathcal{M}_\alpha \rightarrow \mathcal{M}_{\alpha - 1} \rightarrow \cdots \rightarrow \mathcal{M}_{\alpha - k} \rightarrow \cdots, \ k \in \mathbb{N} \]
are isomorphisms, is $b(\alpha - k) \neq 0$ for all integers $k > 0$?

The hypotheses of the proposition above are not satisfied for $\alpha \in \mathbb{N}$, $\alpha \in \frac{1}{4} + \mathbb{N}$ or $\alpha \in -\frac{1}{4} + \mathbb{N}$. But, what happens in these cases?

For $\alpha \in \mathbb{N}$ we have $\mathcal{O}_X[h^{-1}]h^\alpha = \mathcal{O}_X[h^{-1}] = \mathcal{O}_X[*D]$ and it is clear that the image of the map $\varrho_\alpha : \mathcal{M}_\alpha \rightarrow \mathcal{O}_X[h^{-1}]$ is $\mathcal{O}_X$. Also, it is easy to see that this map is not injective.

Let us see what happens for $\alpha = \frac{1}{6}$.

Let us consider the following commutative diagram
\[
\begin{array}{ccc}
\mathcal{M}_{1/6} & \rightarrow & \mathcal{D}_X h^{1/6} \\
\text{Id} \otimes \iota_{1/6} & \downarrow & \downarrow \text{inclusion} \\
\mathcal{M}_{-5/6} & \simeq & \mathcal{D}_X h^{-5/6}.
\end{array}
\]

(1.11) Computation. The natural map $\mathcal{M}_{1/6} \rightarrow \mathcal{M}_{-5/6}$ is neither injective nor surjective: its image is isomorphic to
\[ \mathcal{D}_X/\mathcal{D}_X(\chi - 1, \delta, A) \]
with $A = 9y\partial_y^2\partial_y - 4\partial_y^3 + 180\partial_x^2$, and its kernel is isomorphic to $\mathcal{D}_X/\mathcal{D}_X(x, y)$. Moreover, the above $\mathcal{D}_X$-modules are simple and $\text{ann}_{\mathcal{D}_X} h^{1/6} = \mathcal{D}_X(\chi - 1, \delta, A)$. \[ \square \]

Proof. The map $\text{Id} \otimes \iota_{1/6} : \mathcal{M}_{1/6} \rightarrow \mathcal{M}_{-5/6}$ is given by
\[ F : Q \in \mathcal{D}_X/\mathcal{D}_X(\chi - 1, \delta) \mapsto Qh \in \mathcal{D}_X/\mathcal{D}_X(\chi + 5, \delta). \]
It is an isomorphism outside the origin (where $h$ is a unit or a non-singular curve), so we can concentrate at the origin.

\[ ^{1}\text{In the version of these notes distributed during the course in Vienna there was a mistake: I stated that the surjection } \mathcal{M}_{1/6} \rightarrow \mathcal{D}_X h^{1/6} \text{ was an isomorphism, but this is obviously false. Its kernel coincides with the kernel of } \text{Id} \otimes \iota_{1/6} : \mathcal{M}_{1/6} \rightarrow \mathcal{M}_{-5/6}. \text{ I thank M. Granger for pointing out this error.} \]
Since the symbols $\sigma(\chi)$ and $\sigma(\delta)$ form a regular sequence in $\mathcal{O}_X[\xi, \eta] = \text{gr} \mathcal{D}_X$, we deduce (why?) that $\chi + 5, \delta$ is an involutive basis of the ideal $\mathcal{D}_X(\chi + 5, \delta)$. On the other hand, $\sigma(\chi)$ and $\sigma(\delta)$ vanish at $x = y = 0$ and $\sigma(A)$ does not vanish at $x = y = 0$, so we deduce that $A$ does not belong to the ideal $\mathcal{D}_X(\chi + 5, \delta)$.

From the equality

$$Ah = (3xy\partial_x \partial_y + 2y^2 \partial_y^2 + 6x\partial_x + 12y\partial_y + 12)(\chi + 5) + (-3y^2 \partial_x \partial_y - 2x\partial_y^2 - 9y\partial_x)\delta$$

we deduce that the class of $A$ is a non-zero element in $\text{ker} F$.

So we have a surjective map

$$\mathcal{N} := \mathcal{D}_X(\chi - 1, \delta, 9y\partial_x^2 \partial_y - 4\partial_y^2 + 18\partial_x^2) \to \text{Im} F$$

whose kernel must be supported by the origin (it is an isomorphism on $\mathbb{C}^2 - \{0\}$). Since the characteristic variety of $\mathcal{N}$ does not contain the conormal of the origin we deduce that the preceding map is an isomorphism. Q.E.D.

(1.12) Proposition. For any $\alpha \in \mathbb{C}$ there is a natural isomorphisms $(\mathcal{M}_\alpha)^* \simeq \mathcal{M}_{-\alpha - 1}$, where $^*$ stands for the duality in $D$-module theory. Moreover, the following diagram is commutative

\[
\begin{array}{ccc}
(\mathcal{M}_\alpha)^* & \xrightarrow{(\text{nat.})} & \mathcal{M}_{-\alpha - 1} \\
\downarrow^\text{nat.} & & \downarrow^\text{nat.} \\
(\mathcal{M}_{\alpha - 1})^* & \xrightarrow{(\text{nat.})} & \mathcal{M}_{-\alpha}.
\end{array}
\]

Proof. Compute explicitly the dual $(\mathcal{M}_\alpha)^*$. Q.E.D.

(1.13) Corollary. For all integers $k > 0$, we have $\mathcal{D}_X h^{1/6} = \mathcal{D}_X h^{k+1/6}$ and the natural maps

$$\cdots \to \mathcal{M}_{1/6 + 3} \to \mathcal{M}_{1/6 + 2} \to \mathcal{M}_{1/6 + 1} \to \mathcal{M}_{1/6}$$

are isomorphisms.

The case $\alpha = -\frac{1}{6}$ is similar to the case $\alpha = \frac{1}{6}$.

(1.14) Computation. The map $\text{Id} \otimes t_{-1/6} : \mathcal{M}_{-1/6} \to \mathcal{M}_{-7/6}$ is neither injective nor surjective: its image is isomorphic to

$$\mathcal{D}_X/\mathcal{D}_X(\chi + 1, \delta, 9y\partial_x^2 - 4\partial_y^2)$$

and its kernel is isomorphic to $\mathcal{D}_X/\mathcal{D}_X(x, y)$. Moreover, the above $\mathcal{D}_X$-modules are simple and $\text{ann}_{\mathcal{D}_X} h^{-1/6} = \mathcal{D}_X(\chi + 1, \delta, 9y\partial_x^2 - 4\partial_y^2)$[$^2$]

(1.15) Corollary. For all integers $k > 0$, we have $\mathcal{D}_X h^{-1/6} = \mathcal{D}_X h^{k-1/6}$ and the natural maps

$$\cdots \to \mathcal{M}_{-1/6 + 3} \to \mathcal{M}_{-1/6 + 2} \to \mathcal{M}_{-1/6 + 1} \to \mathcal{M}_{-1/6}$$

$^2$Here a similar remark to the footnote in computation [1.11] proceeds.
are isomorphisms.

(1.16) Remark. (a) The left ideal \( \{Q \in D \mid Qh \in D(\chi + 5, \delta) \} \subset D \) is generated by \( \chi - 1, \delta, A \), and these operators form an involutive basis of the ideal.

(b) The left ideal \( \{Q \in D \mid Qh \in D(\chi + 7, \delta) \} \subset D \) is generated by \( \chi + 1, \delta, A' = 9y\partial^2_x - 4\partial^2_y \), and these operators form an involutive basis of the ideal.

Notice that \( A = A'\partial_y + 18\partial^2_y \).

Summary: For each \( \alpha \in \mathbb{C} \) we have

\[
\text{If } \alpha \notin \mathbb{Z}, \alpha \notin \pm \frac{1}{6} + \mathbb{Z}: \\
\cdots \simeq M_{\alpha+2} \simeq M_{\alpha+1} \simeq M_{\alpha} \simeq M_{\alpha-1} \simeq M_{\alpha-2} \simeq \cdots \simeq \mathcal{O}_X[h^{-1}]h^\alpha = E_\alpha[D]
\]

and \( (M_\alpha)^* \simeq M_\alpha \).

\[
\text{If } \alpha \in \mathbb{Z}: \\
\cdots \simeq M_{\alpha+2} \simeq M_{\alpha+1} \simeq M_\alpha \simeq M_{\alpha-2} \simeq M_{\alpha-1} \simeq \cdots \simeq \mathcal{O}_X[h^{-1}]h^{1/6} = E_{\frac{1}{6}}[D]
\]

and the map \( M_0 \to M_{-1} \) is neither injective nor surjective: its image is isomorphic to \( \mathcal{O}_X \simeq \mathcal{O}_X / \mathcal{O}_X(\partial_x, \partial_y) \) and its kernel is isomorphic to \( \mathcal{D}_X / \mathcal{D}_X(x, y) \). Moreover \( (M_{-1})^* \simeq M_0 \) and the image of the map \( M_0 \to M_{-1} \) is selfdual.

\[
\text{If } \alpha \in \frac{1}{6} + \mathbb{Z}: \\
\cdots \simeq M_{\frac{1}{6}+2} \simeq M_{\frac{1}{6}+1} \simeq M_{\frac{1}{6}} \simeq M_{\frac{1}{6}-1} \simeq M_{\frac{1}{6}-2} \simeq \cdots \simeq \mathcal{O}_X[h^{-1}]h^{1/6} = E_{\frac{1}{6}}[D]
\]

and the map \( M_{\frac{1}{6}} \to M_{\frac{1}{6}-1} \) is neither injective nor surjective: its image is \( \mathcal{D}_X h^{1/6} \simeq \mathcal{D}_X / \mathcal{D}_X(\chi - 1, \delta, 9y\partial^2_x \partial_y - 4\partial^2_y + 18\partial^2_y) \) and its kernel is isomorphic to \( \mathcal{D}_X / \mathcal{D}_X(x, y) \).

\[
\text{If } \alpha \in -\frac{1}{6} + \mathbb{Z}: \\
\cdots \simeq M_{-\frac{1}{6}+2} \simeq M_{-\frac{1}{6}+1} \simeq M_{-\frac{1}{6}-1} \simeq M_{-\frac{1}{6}-2} \simeq \cdots \simeq \mathcal{O}_X[h^{-1}]h^{-1/6} = E_{-\frac{1}{6}}[D]
\]

and the map \( M_{-\frac{1}{6}} \to M_{-\frac{1}{6}-1} \) is neither injective nor surjective: its image is \( \mathcal{D}_X h^{-1/6} \simeq \mathcal{D}_X / \mathcal{D}_X(\chi + 1, \delta, 9y\partial^2_x - 4\partial^2_y) \) and its kernel is isomorphic to \( \mathcal{D}_X / \mathcal{D}_X(x, y) \). Moreover, we have

\[
(\mathcal{D}_X h^{-1/6})^* \simeq \mathcal{D}_X h^{1/6}.
\]

(1.17) Question. Let \( h = 0 \) be a reduced equation defining a germ of hypersurface \( h = 0 \) in \( \mathbb{C}^n \), \( n \geq 2 \), and let \( I(s) \) be the annihilator of \( h^s \) over \( \mathcal{D}_X[s] \). Under which general hypotheses do we have an isomorphism

\[
(\mathcal{D}_X / I(\alpha))^* \simeq \mathcal{D}_X / I(-\alpha - 1)?
\]
2 Local systems on the complement of the cusp

Let us call $U = X - D$ and $j : U \hookrightarrow X$ the corresponding open inclusion. Since the equation of $D$ is quasi-homogeneous, the local topology of $h : \mathbb{C}^2 \to \mathbb{C}$ at the germs at the origin is the “same” as the global one and we can also take $F = h^{-1}(1)$ as its Milnor fiber. The Milnor fibration gives rise to an exact sequence of groups

$$1 \to L = \pi_1(F) \to G = \pi_1(U) \to \pi_1(\mathbb{C}^*, 1) \to 1$$

for convenable base points. The group $L$ is a free group of rank 2 and the group $\pi_1(\mathbb{C}^*, 1)$ is isomorphic to $(\mathbb{Z}, +)$ with “positive” generator $\delta$. The restriction of the Milnor fibration to a big enough sphere centered at the origin gives rise to an exact sequence

$$1 \to L^\partial = \pi_1(F^\partial) \to G^\partial = \pi_1(U^\partial) = L^\partial \times \pi_1(\mathbb{C}^*, 1) \to \pi_1(\mathbb{C}^*, 1) \to 1.$$

Since the Milnor fibration restricted to the bord extends to the origin, the above sequence splits and there is a natural scission $\sigma : \pi_1(\mathbb{C}^*, 1) \to G$ and so we have that $G = L \ltimes \pi_1(\mathbb{C}^*, 1)$ with $\tau : \pi_1(\mathbb{C}^*, 1) \to \text{Aut}(L)$ given by $\tau(\delta)(\beta) = \sigma(\delta)\beta\sigma(\delta)^{-1}$. Actually, the automorphism $\tau(\delta) : L \to L$ is induced by the geometric monodromy $T : F \to F$ associated with the Milnor fibration. The bord of the Milnor fibration $F^\partial$ is a cercle and its fundamental group is infinite cyclic with positive generator $\gamma$. Since the Milnor fibration restricted to the bord is trivial we have $\sigma(\delta)\gamma\sigma(\delta)^{-1} = \gamma$. To simplify, we identify $\pi_1(\mathbb{C}^*, 1)$ with its image by $\sigma$.

(2.1) Lemma. In the case of the cusp with equation $h = x^2 - y^3$ there is a basis of $L$ with two elements $a, b$ such that

$$\delta a b^{-1} = b^{-1}, \quad \delta b a^{-1} = b a$$

and so $G$ has a presentation

$$G = \langle a, b, \delta; \delta a b^{-1} = b^{-1}, \delta b a^{-1} = b a \rangle.$$

(2.2) Definition. For any complex number $z \in \mathbb{C}^*$ let us call $\mathcal{L}_z$ the local system on $U$ of rank 1 given by the representation $\varrho_z : G \to GL(1, \mathbb{C}) = \mathbb{C}^*$ with

$$\varrho_z(\delta) = z, \quad \varrho_z(a) = \varrho_z(b) = 1.$$

It can be also described as the inverse image by $h$ of the local system on $\mathbb{C}^*$ given by the representation

$$m \in \mathbb{Z} \equiv \pi_1(\mathbb{C}^*, 1) \mapsto z^m \in \mathbb{C}^*.$$

(2.3) Proposition. With the above notations, there are natural isomorphisms of local systems on $U$:

1. $\mathcal{L}_z \otimes_{\mathbb{C}} \mathcal{L}_w \simeq \mathcal{L}_{zw}$, $z, w \in \mathbb{C}^*$. 

7
2. \( \mathcal{D}_1 \simeq \mathbb{C}_U \).

3. \((\mathcal{D}_z)^* = \text{Hom}_{\mathbb{C}_U} (\mathcal{D}_z, \mathbb{C}_U) \simeq \mathcal{D}_{1/z} \).

(2.4) Proposition. For any local system \( \mathcal{L} \) on \( U \) of rank 1 there is a unique \( z \in \mathbb{C}^* \) such that \( \mathcal{L} \simeq \mathcal{L}_z \).

(2.5) Theorem. Let \( z \in \mathbb{C}^* \) be a non-zero constant. The following properties hold:

- If \( z \neq 1, e^{2\pi i z} : \text{DR}(\mathcal{M}_n) \simeq R_j \mathcal{L}_z \simeq j_* \mathcal{L}_z \simeq j_* \mathcal{L}_z \) for any \( \alpha \in \mathbb{C} \) such that \( e^{2\pi i \alpha} = z \).
- If \( z = 1, \text{DR}(\mathcal{M}_-k) \simeq R_j \mathcal{C}_U \) for any integer \( k \geq 1 \), \( \text{DR}(\mathcal{M}_k) \simeq j_* \mathcal{C}_U \) for any integer \( k \geq 0 \) and \( \text{DR}(\mathcal{O}_X) = \mathbb{C}_X \simeq j_* \mathcal{C}_U \).
- If \( z = e^{2\pi i z} : \text{DR}(\mathcal{M}_{1-k}) \simeq R_j \mathcal{L}_z \) for any integer \( k \geq 1 \), \( \text{DR}(\mathcal{M}_{1+k}) \simeq j_* \mathcal{L}_z \) for any integer \( k \geq 0 \) and \( \text{DR}(\mathcal{O}_X h^{1/6}) \simeq j_* \mathcal{L}_z \).
- If \( z = e^{-2\pi i z} : \text{DR}(\mathcal{M}_{-1-k}) \simeq R_j \mathcal{L}_z \) for any integer \( k \geq 1 \), \( \text{DR}(\mathcal{M}_{-1+k}) \simeq j_* \mathcal{L}_z \) for any integer \( k \geq 0 \) and \( \text{DR}(\mathcal{O}_X h^{-1/6}) \simeq j_* \mathcal{L}_z \).

(2.6) Remark. Let us call \( \mathcal{MC}_1 \) the set of isomorphism classes of integrable logarithmic connections with respect to \( D \) of rank 1, \( \mathcal{C}_{1} \) the set of isomorphism classes of regular meromorphic connections with respect to \( D \) of rank 1 and \( \mathcal{D}_1 \) the set of isomorphism classes of local systems on \( U \) of rank 1. There are natural bijective maps \( \mathcal{MC}_1 \leftrightarrow \mathbb{C}, \mathcal{MC}_1 \leftrightarrow \mathbb{C}/(\mathbb{Z}2\pi i), \mathcal{D}_1 \leftrightarrow \mathbb{C}^* \) in such a way that the diagram \( \mathcal{MC}_1 \rightarrow \mathcal{MC}_1 \rightarrow \mathcal{D}_1 \), where the first map is the localization along \( D \) and the second map is “taking the horizontal sections on \( U \)," corresponds to

\[
\mathbb{C} \xrightarrow{\text{nat.}} \mathbb{C}/(\mathbb{Z}2\pi i) \xrightarrow{\mathbb{Z} \mapsto e^{2\pi i \alpha}} \mathbb{C}^*.
\]

Moreover, the duality on \( \mathcal{MC}_1 \) corresponds to the map \( \alpha \in \mathbb{C} \mapsto -\alpha \in \mathbb{C} \) and the duality on \( \mathcal{D}_1 \) corresponds to the map \( z \in \mathbb{C}^* \mapsto z^{-1} \in \mathbb{C}^* \).

3 Local systems of rank two on the complement of the cusp

We consider again \( h : X = \mathbb{C}^2 \rightarrow \mathbb{C}, h(x,y) = x^2 - y^3, D = h^{-1}(0), j : U = X - D \hookrightarrow X \).

(3.1) Theorem. (MacPherson-Vilonen, Deligne, Verdier; 1982-83) A perverse sheaf \( \mathbf{K} \) on \( X \) stratified by \( \{0\}, D - \{0\}, U \) is determined by

\[
(\mathcal{D}, \mathbf{F}, u : R\psi_h \mathcal{D} \rightarrow \mathbf{F}, v : \mathbf{F} \rightarrow R\psi_h \mathcal{D})
\]

with \( \mathcal{D} = j^* \mathbf{K} \) a local system on \( U \), \( \mathbf{F} = \phi_0 \mathbf{K} \) a perverse sheaf on \( D \) (stratified w.r.t. \( \{0\}, D - \{0\} \)) and \( u, v \) maps of perverse sheaves such that

\[
\text{Id} + v \circ u = T_{\mathcal{D}} : R\psi_h \mathcal{D} \rightarrow R\psi_h \mathcal{D}.
\]
But \((D, 0) \simeq (\mathbb{C}, 0)\) and perverse sheaves on \(D\) stratified by \(\{0\}, D - \{0\}, U\) are well known.

(3.2) Theorem. (The computation of \(R\psi_h \mathcal{L}\)) Assume that the local system \(\mathcal{L}\) on \(U\) is given by a representation \(G = \pi_1(U) \to \text{GL}(E)\). The perverse sheaf \(R\psi_h \mathcal{L}\) on \(D\) is given by the diagram

\[
(E, \text{Hom}_{\mathbb{C}[L]}(I(L), E), U, V)
\]

with

\[
U : E \to \text{Hom}_{\mathbb{C}[L]}(I(L), E), \quad U(e)(g) = ge,
\]

\[
V : \text{Hom}_{\mathbb{C}[L]}(I(L), E) \to E, \quad V(\varphi) = \varphi(\gamma - 1).
\]

Moreover, the “monodromy automorphism” \(T_\mathcal{L} : R\psi_h \mathcal{L} \to R\psi_h \mathcal{L}\) is given by:

\[
t_1 : E \to E, \quad t_2 : \text{Hom}_{\mathbb{C}[L]}(I(L), E) \to \text{Hom}_{\mathbb{C}[L]}(I(L), E)
\]

with \(t_1(e) = \delta^{-1} e, t_2(\varphi)(g) = \delta^{-1} \varphi(\delta g \delta^{-1}),\) where \(I(L)\) is the augmentation ideal of \(L\), i.e. the kernel of \(\mathbb{C}[L] \to \mathbb{C}\).

(3.3) Corollary. (Explicit description; \([2]\)) The perverse sheaf \(\mathcal{K}\) on \(X\) stratified by \(\{0\}, D - \{0\}, U\) and determined by \((\mathcal{L}, \mathcal{F}, u : R\psi_h \mathcal{L} \to \mathcal{F}, v : \mathcal{F} \to R\psi_h \mathcal{L})\) is explicitly described by \(\mathcal{L}\) is given by a complex representation \(G = \pi_1(U) \to \text{GL}(E)\).

\(\mathcal{F}\) is given by a diagram of vector spaces \((C_1, C_2; p : C_1 \to C_2, q : C_2 \to C_1)\) with \(\text{Id} + q \circ p\) is \(\simeq\).

\((R\psi_h \mathcal{L}, T_\mathcal{L})\) is given by:

\[
(E, \text{Hom}_{\mathbb{C}[L]}(I(L), E), U, V) \cap (t_1, t_2) \text{ with }
\]

\[
U : E \to \text{Hom}_{\mathbb{C}[L]}(I(L), E), \quad U(e)(g) = ge,
\]

\[
V : \text{Hom}_{\mathbb{C}[L]}(I(L), E) \to E, \quad V(\varphi) = \varphi(\gamma - 1),
\]

and \(t_1(e) = \delta^{-1} e, t_2(\varphi)(g) = \delta^{-1} \varphi(\delta g \delta^{-1})\).

(3.4) Corollary. (A formula for the characteristic cycle) If our perverse sheaf \(\mathcal{K}\) on \(X\) is given by \(G \to \text{GL}(E), (C_1, C_2; p : C_1 \to C_2, q : C_2 \to C_1)\) and \(u_1 : E \to C_1, v_1 : C_1 \to E, u_2 : \text{Hom}_{\mathbb{C}[L]}(I(L), E) \to C_2, v_2 : C_2 \to \text{Hom}_{\mathbb{C}[L]}(I(L), E)\) with the corresponding commutativity conditions and \(\text{Id} + v_1 \circ u_1 = t_1, \text{Id} + v_2 \circ u_2 = t_2\), then:

\[
\text{CC}(\mathcal{K}) = m_2 T^*_X(X) + m_1 T^*_{D_{nu}}(X) + m_0 T^*_h(X),
\]

\(m_2 = \dim E, m_1 = \dim C_1, m_0 = \mu \dim E + (1 - e) \dim C_1 + \dim C_2,\) where \(\mu\) is the Milnor number of \(h : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)\) and \(e\) is the multiplicity of \((D, 0)\).

The results above are true for any germ of irreducible plane curve. In the case of the cusp above we have:
\[ L \leftrightarrow \rho : G = \langle a, b, \delta; \delta a \delta^{-1} = b^{-1}, \delta b \delta^{-1} = ba \rangle \to \text{GL}(E) : \]
\[ \rho(a) = A, \quad \rho(b) = B, \quad \rho(\delta) = \Delta. \]

- \( R\psi_h \mathcal{L} \) is the perverse sheaf on \( D \) given by:

\[
(E, E^2, \begin{pmatrix} A - I \\ B - I \end{pmatrix}, (B - BAB^{-1} A^{-1}, I - BAB^{-1}))
\]

and \( T\mathcal{L} \) is given by:

\[
t_1 = \Delta^{-1}, \quad t_2 = \begin{pmatrix} 0 & -\Delta^{-1}B^{-1} \\ \Delta^{-1}B^{-1} & \Delta^{-1} \end{pmatrix}.
\]

(3.5) Corollary. (Description of intersection complexes) Assume that \( L \) is a local system on \( U = X - D \). The intersection complex \( j_!^* L \) is the perverse sheaf given by

\[
(\mathcal{L}, \text{Im}(T\mathcal{L} - 1), T\mathcal{L} - 1, \text{inclusion}).
\]

Moreover if \( \mathcal{L} \) is associated with the representation \( G \to \text{GL}(E) \), then

\[
\text{CC}(j_!^* L) = m_2 T_\mathcal{L}^*(X) + m_1 T_{D_{reg}}^*(X) + m_0 T_0^*(X),
\]

where

\[
m_2 = \dim E, \quad m_1 = \text{rank}(t_1 - 1), \quad m_0 = \text{rank}(t_2 - 1) - \mu m_2 + (\text{mult}_0(D) - 1)m_1,
\]

with \( \mu = 2 \) and \( \text{mult}_0(D) = 2 \).

Intersection complexes associated with local systems of rank 1 on the complement of the cusp

- For \( z \in \mathbb{C}^* \), let \( \mathcal{L}_z \) be the local system on \( U \) of rank 1 given by \( \varrho_z : G \to \text{GL}(1, \mathbb{C}) = \mathbb{C}^* \), \( \varrho_z(\delta) = z, \varrho_z(a) = \varrho_z(b) = 1 \).

- \( t_1 = z^{-1}, \quad t_2 = \begin{pmatrix} 0 & -z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix} \)

- \( \text{CC}(j_!^* \mathcal{L}_z) = T_\mathcal{L}^*(X) + m_1 T_{D_{reg}}^*(X) + m_0 T_0^*(X) \).

- If \( z = 1 \) then \( \mathcal{L}_z = \mathbb{C}U \) and \( j_!^* \mathcal{L}_z = \mathbb{C}X \), \( m_1 = m_0 = 0 \).

- If \( z \neq 1 \) then \( m_1 = 1 \) and

\[
\begin{cases} 
\text{if } z^2 - z + 1 = 0 \text{ then } m_0 = 0 \\
\text{if } z^2 - z + 1 \neq 0 \text{ then } m_0 = 1.
\end{cases}
\]

- \( z^2 - z + 1 = 0 \leftrightarrow z = e^{\pm \frac{2\pi i}{3}} \).

Some examples of local systems of rank 2 on the complement of the cusp

- For \( s, t \in \mathbb{C}^* \) let \( \mathcal{L}_{s,t} \) be the local system associated with the representation \( \varrho_{s,t} : G \to \text{GL}(2, \mathbb{C}) : \varrho_{s,t}(a) = \varrho_{s,t}(b) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^2 \end{pmatrix}, \varrho_{s,t}(\delta) = \begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix}, \theta = e^{\frac{2\pi i}{3}} \).
Some examples of integrable logarithmic connections of rank 2

For any constants \( \lambda, e \in \mathbb{C} \) let us consider the integrable logarithmic connection \( \mathcal{E}_{\lambda, e} \) whose underlying \( \mathcal{O}_X \)-module is \( \mathcal{O}_X^\lambda \) and the actions of \( \chi \) and \( \delta \) with respect to the basis \( e_1 = (1,0), e_2 = (0,1) \) are given by:

\[
\chi \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \delta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ ey & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.
\]

Let us call

\[
A_{\lambda} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + 1 \end{pmatrix}, \quad B_{e} = \begin{pmatrix} 0 & 1 \\ ey & 0 \end{pmatrix}.
\]

It is clear that \( \mathcal{E}_{\lambda, e} = \mathcal{F}_X e_1 \) and that \( \text{ann}_{\mathcal{F}_X} e_1 = \mathcal{F}_X (\chi - \lambda, \delta^2 - ey) \).

We call \( \mathcal{M}_{\lambda, e} = \mathcal{D}_X \otimes_{\mathcal{F}_X} \mathcal{E}_{\lambda, e} = \mathcal{D}_X / \mathcal{D}_X (\chi - \lambda, \delta^2 - ey) \).

(4.1) PROPOSITION. With the above notations, there are natural isomorphisms of left \( \mathcal{F}_X \)-modules:

1) \( \mathcal{E}_{\lambda, e} \otimes_{\mathcal{D}_X} \mathcal{E}_{\alpha} \simeq \mathcal{E}_{\lambda+\alpha, e}, \alpha, \lambda, e \in \mathbb{C}. \)

2) \( \mathcal{E}_{\lambda, e}^* = \text{Hom}_{\mathcal{D}_X} (\mathcal{E}_{\lambda, e}, \mathcal{O}_X) \simeq \mathcal{E}_{-\lambda-1, e}. \)

PROOF. 2) \( (\chi e_1^*)(e_1) = \chi(e_1^*(e_1)) - e_1^*(\chi e_1) = -\lambda, (\chi e_1^*)(e_2) = \chi(e_1^*(e_2)) - e_1^*(\chi e_2) = 0 \). So \( \chi e_1^* = -\lambda e_1^* \). In a similar way we find \( \chi e_2^* = -(\lambda + 1)e_2^* \).

\( (\delta e_1^*)(e_1) = 0, (\delta e_1^*)(e_2) = 0 \). So \( \delta e_1^* = e_1^* \). We also find \( \delta e_2^* = e_2^* \).

The \( \mathcal{D}_X \)-linear map \( \mathcal{E}_{-\lambda-1, e} \to (\mathcal{E}_{\lambda, e})^* \) sending \( e_1 \mapsto e_2^*, e_2 \mapsto -e_1^* \) is an isomorphism of \( \mathcal{F}_X \)-modules.

Q.E.D.

We have natural maps \( \mathcal{M}_{\lambda, e} \to \mathcal{M}_{\lambda-6, e} \) induced by the inclusion \( \mathcal{E}_0 \to \mathcal{E}_{-1} \).

(4.2) PROPOSITION. With the above notations, there is a natural isomorphism \( \mathcal{M}_{\lambda, e}^* \simeq \mathcal{M}_{-\lambda-7, e} \), where \( * \) stands for the duality in \( D \)-module theory. Moreover, the following diagram is commutative

\[
\begin{array}{ccc}
(M_{\lambda, e})^* \overset{(\text{nat.})}{\cong} M_{-\lambda-7, e} \\
\downarrow \text{nat.}^* & & \uparrow \text{nat.} \\
(M_{\lambda-6, e})^* \overset{(\text{nat.})}{\cong} M_{-\lambda-1, e}.
\end{array}
\]
Proof. Compute explicitly the dual \((\mathcal{M}_{\lambda,e})^*\).

Let us denote by \(\Phi : \mathcal{F} \rightarrow \mathcal{F}\) the automorphism of \(\mathcal{F}\) such that \(\Phi\) is the identity on \(\mathcal{O}\) and \(\Phi(\varepsilon) = \varepsilon - \frac{\varepsilon(h)}{h}\) for any logarithmic derivation \(\varepsilon\). In particular
\[
\Phi(\delta) = \delta, \quad \Phi(\chi) = \chi - 6.
\]

It is clear that for any logarithmic differential operator \(Q\), one has
\[
(Qe_1) \otimes h^{-1} = \Phi(Q)(e_1 \otimes h^{-1})
\]
in \(E_{\lambda,e}(D) = E_{\lambda,e} \otimes_{\mathcal{O}} \mathcal{O}(D)\), and so
\[
(Qe_1) \otimes h^{-k} = \Phi^k(Q)(e_1 \otimes h^{-k})
\]
in \(E_{\lambda,e}(kD) = E_{\lambda,e} \otimes_{\mathcal{O}} \mathcal{O}(kD)\) for any integer \(k \in \mathbb{Z}\).

We have
\[
\left( \begin{array}{c}
\partial_x \\
\partial_y
\end{array} \right) = \frac{1}{6h} \left( \begin{array}{cc}
2x & -2y \\
-3y^2 & 3x
\end{array} \right) \left( \begin{array}{c}
\chi \\
\delta
\end{array} \right).
\]

The meromorphic connection \(E_{\lambda,e}[\ast D]\) is given, as left \(\mathcal{D}[\ast D]\)-module, by:
\[
\partial_x \left( \begin{array}{c}
e_1 \\
e_2
\end{array} \right) = \frac{1}{6h} \left( 2xA_\lambda - 2yB_\varepsilon \right) \left( \begin{array}{c}
e_1 \\
e_2
\end{array} \right) = \left( \begin{array}{cc}
\frac{\lambda x}{2h} & -\frac{y}{2h} \\
-\frac{y}{2h} & \frac{\lambda y}{2h}
\end{array} \right) \left( \begin{array}{c}
e_1 \\
e_2
\end{array} \right),
\]
\[
\partial_y \left( \begin{array}{c}
e_1 \\
e_2
\end{array} \right) = \frac{1}{6h} \left( -3y^2A_\lambda + 3xB_\varepsilon \right) \left( \begin{array}{c}
e_1 \\
e_2
\end{array} \right) = \left( \begin{array}{cc}
\frac{-\lambda y^2}{2h} & \frac{\lambda x}{2h} \\
\frac{\lambda x}{2h} & \frac{-\lambda y}{2h}
\end{array} \right) \left( \begin{array}{c}
e_1 \\
e_2
\end{array} \right).
\]

(4.3) Definition. We consider the left \(\mathcal{F}[s]\)-module \(E_{\lambda,e}[s|h^s]\) where the action of \(\chi\) and \(\delta\) are given by:
\[
\chi \left( \begin{array}{c}
e_1 h^s \\
e_2 h^s
\end{array} \right) = \left( \begin{array}{cc}
\lambda + 6s & 0 \\
0 & \lambda + 6s + 1
\end{array} \right) \left( \begin{array}{c}
e_1 h^s \\
e_2 h^s
\end{array} \right), \quad \delta \left( \begin{array}{c}
e_1 h^s \\
e_2 h^s
\end{array} \right) = \left( \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right) \left( \begin{array}{c}
e_1 h^s \\
e_2 h^s
\end{array} \right).
\]

It is a sub-\(\mathcal{F}[s]\)-module of the Bernstein module \(E_{\lambda,e}[s,h^{-1}]h^s\).

(4.4) Proposition. The left \(\mathcal{F}[s]\)-module \(E_{\lambda,e}[s|h^s]\) is generated by \(e_1 h^s\) and \(\text{ann}_\mathcal{F}[s](e_1 h^s) = \mathcal{F}[s](\chi - \lambda - 6s, \delta^2 - ey)\).

(4.5) Exercise. Prove the above proposition.

(4.6) Proposition. The annihilator of \(e_1 h^s \in E_{\lambda,e}[s,h^{-1}]h^s\) over \(\mathcal{D}[s]\) is the left ideal generated by \(\chi - \lambda - 6s\) and \(\delta^2 - ey\), or in other words, the canonical map
\[
Q \otimes q \in \mathcal{D}[s] \otimes_{\mathcal{F}[s]} (E_{\lambda,e}[s|h^s]) \mapsto Qq \in \mathcal{D}[s] : (e_1 h^s) \subset E_{\lambda,e}[s,h^{-1}]h^s
\]
is an isomorphism.

(4.7) Exercise. Prove the above proposition.
(4.8) Computation. (Done with Macaulay 2!) Let \( f \in \mathbb{C} \) be such that \( e = f^2 - f \). The polynomial
\[
    b_{\lambda,f}(s) = \left( s + \frac{\lambda + f + 6}{6} \right) \left( s + \frac{\lambda - f + 7}{6} \right) \left( s + \frac{\lambda + 5}{6} \right) \left( s + \frac{\lambda + 8}{6} \right)
\]
belongs to the ideal \( \mathcal{D}(h, \chi - \lambda - 6s, \delta^2 - ey) \). More precisely, there are \( R(s), C(s), D(s) \in \mathbb{C}[x, y, \lambda, f, s][\partial_x, \partial_y] \) such that
\[
    b_{\lambda,f}(s) = R(s)h + C(s)(\chi - \lambda - 6s) + D(s)(\delta^2 - ey),
\]
where \( \lambda, f, s \) are parameters and \( e = f^2 - f \).  

(4.9) Remark. Notice the following facts:

1. For \( \lambda, f \in \mathbb{C} \), the polynomial \( b_{\lambda,f}(s) \in \mathbb{C}[s] \) does not depend on \( f \), it only depends on \( \lambda \) and \( e = f^2 - f \). If we define \( \sigma(f) = -f + 1 \), we have \( \sigma(f)^2 - \sigma(f) = f^2 - f = e \), and so \( f \) and \( \sigma(f) \) are the solutions of \( x^2 - x = e \). We have \( b_{\lambda,f}(s) = b_{\lambda,\sigma(f)}(s) \), and so we can define \( b_{\lambda,e}(s) := b_{\lambda,f}(s) \) for \( f \) such that \( e = f^2 - f \).
2. We have \( b_{\lambda,e}(s) = b_{\lambda,e}(-s - 2) \), with \( \lambda^* = -\lambda - 1 \).

(4.10) Question. Can we always, or at least in “many” other examples, to obtain a parametric functional equation of the type \( \square \) for other families of integrable logarithmic connections, eventually by doing a ramification (as \( f \mapsto e = f^2 - f \)?)

(4.11) Question. Under which conditions on the (free) divisor and on the integrable logarithmic connection \( \mathcal{E} \) we have the following equality
\[
    b_{\mathcal{E},p}(s) = \pm b_{\mathcal{E},p}(-s - 2)
\]
where \( b_{\mathcal{E},p}(s) \) denotes the Bernstein-Sato polynomial (at a point \( p \)) of \( \mathcal{E} \) (see \( \square \), lemma 3.4, remark 3.5). This question is obviously related with \( \square \).

(4.12) Proposition. For any \( \lambda, e \in \mathbb{C} \) such that \( b_{\lambda,e}(-i) \neq 0 \) for all integers \( i \geq 1 \), or equivalently, \( (\lambda + f + 6)/6, (\lambda - f + 7)/6, (\lambda + 5)/6, (\lambda + 8)/6 \notin \mathbb{N}_+ \) (with \( f^2 - f = e \)) the natural maps
\[
    P \otimes g \in \mathcal{M}_{\lambda,e} = \mathcal{D}_X \otimes_{\mathcal{D}_X} \mathcal{E}_{\lambda,e} \mapsto P(g) \in \mathcal{E}_{\lambda,e}[*D]
\]
and
\[
    \mathcal{M}_{\lambda,e} \to \mathcal{M}_{\lambda-6,e} \to \mathcal{M}_{\lambda-12,e} \to \mathcal{M}_{\lambda-18,e} \to \cdots
\]
are isomorphisms.

Proof. We only treat the behaviour at the origin. The details for the other points are left to the reader. Let us first consider the map \( \mathcal{M}_{\lambda,e} \to \mathcal{E}_{\lambda,e}[*D] \). For the surjectivity, it is enough to consider the case of \( e_1 h^{-k} \) for \( k > 0 \), since for any section \( g \) of \( \mathcal{E}_{\lambda,e} \) there is a logarithmic differential operator \( Q \) such that \( g = Qe_1 \) and so \( gh^{-k} = (Qe_1)h^{-k} = \Phi^k(Q)(e_1 h^{-k}) \).
From (4) we know that \((b_{\lambda,f}(-1) - R(-1)h)(e_1h^{-1}) = 0\) and so
\[
e_1h^{-1} = \left( \frac{R(-1)}{b_{\lambda,f}(-1)} \right) e_1
\]
is in the image of the map \(\mathcal{M}_{\lambda,e} \to \mathcal{E}_{\lambda,e}[*D]\). In a similar way we prove that for any integer \(k > 0\) we have
\[
e_1h^{-k} = \left( \frac{R(-k)R(-k+1)\cdots R(-1)}{b_{\lambda,f}(-k)b_{\lambda,f}(-k+1)\cdots b_{\lambda,f}(-1)} \right) e_1
\]
and so \(e_1h^{-k}\) is also in the image of the map \(\mathcal{M}_{\lambda,e} \to \mathcal{E}_{\lambda,e}[*D]\).

The proof of the injectivity is similar to the proof of proposition (1.9) (Exercise: Prove it!; Hint: see the proof of theorem 3.1 in [1]). Q.E.D.

(4.13) Proposition. For any \(\lambda,f \in \mathbb{C}\), there is a \(k_0 \geq 0\) such that for each \(k \geq k_0\) the natural map
\[
P \otimes g \in \mathcal{M}_{\lambda-6k,e} = \mathcal{D}_X \otimes_{\mathcal{F}_X} \mathcal{E}_{\lambda,e}(kD) \mapsto P(g) \in \mathcal{E}_{\lambda,e}[*D]
\]
\((e = f^2 - f)\) is an isomorphism.

Summary (see [2]): Let \(\mathcal{D}_{\lambda,e}\) the local system of rank 2 on \(U\) of the horizontal sections of \(\mathcal{E}_{\lambda,e}\)

(a) For any \(\lambda,f \in \mathbb{C}\) such that \((\lambda+f)/6, (\lambda-f+1)/6, (\lambda-1)/6, (\lambda+2)/6 \notin \mathbb{N}_+\), the natural map
\[
P \otimes g \in \mathcal{M}_{\lambda-6k,e} = \mathcal{D}_X \otimes_{\mathcal{F}_X} \mathcal{E}_{\lambda,e}(D) \mapsto P(g) \in \mathcal{E}_{\lambda,e}[*D]
\]
is an isomorphism. In particular \(R_j\mathcal{D}_{\lambda,e} \simeq \text{DR}(\mathcal{M}_{\lambda-6,k})\).

(b) For any \(\lambda,f \in \mathbb{C}\) such that \((-\lambda+f-1)/6, (-\lambda-f)/6, (-\lambda-2)/6, (-\lambda+1)/6 \notin \mathbb{N}_+\), the natural map
\[
P \otimes g \in \mathcal{D}_X \otimes_{\mathcal{F}_X} \mathcal{E}_{\lambda-7,e} = \mathcal{M}_{\lambda-7,e} \mapsto P(g) \in \mathcal{E}_{\lambda,e}[*D]
\]
is an isomorphism. In particular \(R_j\mathcal{D}_{\lambda,e}^* \simeq \text{DR}(\mathcal{M}_{\lambda-7,e})\).

(c) With the same hypotheses as in (b), the induced map
\[
((\mathcal{E}_{\lambda,e})^*[\mathcal{D}])^* \mapsto (\mathcal{M}_{\lambda-7,e})^* \simeq \mathcal{M}_{\lambda,e}
\]
is an isomorphism. In particular \(j^*\mathcal{D}_{\lambda,e} \simeq \text{DR}(\mathcal{M}_{\lambda,e})\).

(d) If \(\lambda,f \in \mathbb{C}\) satisfies the hypotheses in (a) and (b), then the regular holonomic \(\mathcal{D}_X\)-module computing the intersection complex \(j_*\mathcal{D}_{\lambda,e}\) is the image of \(\mathcal{M}_{\lambda,e} \to \mathcal{M}_{\lambda-6,e}\).

(4.14) Proposition. There is no nontrivial map of integrable logarithmic connections (ILC for short) \(\mathcal{E}_\alpha \to \mathcal{E}_{\lambda,e}\) if \(e \notin 6\mathbb{N}\).
Proof. We can obviously work at the level of the stalks at the origin. Let \( \varphi : \mathcal{E}_{\alpha} \to \mathcal{E}_{\lambda,e} \) be a map of ILC and let us write \( \varphi(h^\alpha) = (a, b), a, b \in \theta \). Since \( \varphi \) is \( \mathcal{F} \)-linear, the following equalities hold:

\[
6a(a, b) = \varphi(6\alpha h^\alpha) = \varphi(\chi h^\alpha) = \chi \cdot (a, b) = \cdots = ((\chi + \lambda)(a), (\chi + \lambda + 1)(b)),
\]

0 = \( \varphi(\delta h^\alpha) = \delta \cdot (a, b) = \cdots = (\delta(a) + e y b, a + \delta(b)) \).

In particular

\[
\chi(a) = (6\alpha - \lambda)a, \quad \chi(b) = (6\alpha - \lambda - 1)b,
\]

and \( \delta(a) + e y b = 0, \delta(b) = -a \).

Assume that \( \varphi \neq 0 \). Then \( a, b \neq 0 \) and so \( 6\alpha - \lambda, 6\alpha - \lambda - 1 \) should be weights in \( W := 2\mathbb{N} + 3\mathbb{N} = \{0, 2, 3, 4, 5, \ldots \} \). Let us call \( w = 6\alpha - \lambda - 1 \geq 2 \)

and \( b \) will be quasi-homogeneous of weight \( w \). We have \( \delta^2(b) = -\delta(a) = e y b \), but this contradicts proposition (4.16). Q.E.D.

(4.15) Proposition. Let \( \mathcal{L}_{\lambda,e} \) the local system of rank 2 on \( U \) of the horizontal sections of \( \mathcal{E}_{\lambda,e} \). Then, \( \mathcal{L}_{\lambda,e} \) is irreducible if \( e \notin 6\mathbb{N} \).

Proof. From [3], theorem 10.3-2, we know that the de Rham functor establishes an equivalence of abelian categories between the category of regular meromorphic connections along \( D \) and the category of local systems on \( U \). The regular meromorphic connection corresponding to \( \mathcal{L}_{\lambda,e} \) is just the localization \( \mathcal{E}_{\lambda,e} \) \( \mathcal{H} \). Assume that the local system is not irreducible. Then, there is a local system of rank 1, necessarily of the type \( \mathcal{L}_{z, z} \), \( z \in \mathbb{C}^* \), and a non-trivial (and so injective) map \( \mathcal{L}_z \to \mathcal{L}_{\lambda,e} \).

Let \( \alpha \in \mathbb{C} \) be such that \( e^{2\pi i \alpha} = z \) and so \( \text{DR}(\mathcal{E}_{\alpha} \mathcal{H}) \approx R_{\mathcal{J}} \mathcal{L}_z \). We deduce the existence of a non-trivial (and so injective) map of meromorphic connections \( \mathcal{E}_{\alpha} \mathcal{H} \to \mathcal{E}_{\lambda,e} \mathcal{H} \), and so the existence of a non-trivial (and so injective) map of integrable logarithmic connections \( \mathcal{E}_{\alpha} \to \mathcal{E}_{\lambda,e}(kD) = \mathcal{E}_{\lambda - 6k,e} \) for some \( k >> 0 \), but this contradicts proposition (4.14). Q.E.D.

(4.16) Proposition. For any weight \( w \in 2\mathbb{N} + 3\mathbb{N} \), let us call \( \mathcal{P}_w \) the vector space of quasi-homogeneous polynomials in \( \mathbb{C}[x, y] \) of weight \( w \) with respect to weights \( w(x) = 3, w(y) = 2 \). For any constant \( e \in \mathbb{C} \) such that \( e \notin 6\mathbb{N} \), the map

\[
\delta^2 - e y : \mathcal{P}_w \to \mathcal{P}_{w+2}
\]

is injective.

Proof. For \( w = 0 \) we have \( \mathcal{P}_w = \langle 1 \rangle \) and the result is clear.

We have \( \delta^2(x) = 12xy, \delta^2(y) = 6y^2, \delta^2(xy) = 30xy^2 \) and if \( n \geq 2 \) then

\[
\delta^2(y^n) = 4n(n - 1)y^{n-2}h + 2n(2n + 1)y^{n+1},
\]

\[
\delta^2(xy^n) = 4n(n - 1)xy^{n-2}h + (2n + 3)(2n + 4)xy^{n+1}.
\]

For \( 2 \leq w \leq 6 \) we have \( \mathcal{P}_2 = \langle y \rangle, \mathcal{P}_3 = \langle x \rangle, \mathcal{P}_4 = \langle y^2 \rangle, \mathcal{P}_5 = \langle xy \rangle, \mathcal{P}_6 = \langle x^2, y^3 \rangle = \langle h, y^3 \rangle \). Since:

\[
(\delta^2 - e y)(y) = (6 - e)y^2, \delta^2 - e y : \mathcal{P}_2 \to \mathcal{P}_4 \text{ is injective if } e \neq 6.
\]

\[
(\delta^2 - e y)(x) = (12 - e)xy, \delta^2 - e y : \mathcal{P}_3 \to \mathcal{P}_5 \text{ is injective if } e \neq 12.
\]
\[(\delta^2 - cy)(y^2) = 8h + (20 - e)y^3, \delta^2 - ey : \mathcal{P}_4 \to \mathcal{P}_0 \text{ is injective } \forall e \in \mathbb{C}.
\]

\[(\delta^2 - cy)(xy) = (30 - e)xy^2, \delta^2 - ey : \mathcal{P}_5 \to \mathcal{P}_7 \text{ is injective if } e \neq 30.
\]

\[(\delta^2 - ey)(\gamma_1 h + \gamma_0 y^3) = (24\gamma_0 - e\gamma_1)yh + (42 - e)\gamma_0 y^4, \delta^2 - ey : \mathcal{P}_6 \to \mathcal{P}_8 \text{ is injective if } e \neq 42, 0.
\]

Take \(w \geq 7\) and write \(w = 6k + w'\) with \(0 \leq w' < 6\). For any \(a \in \mathcal{P}_w\) we can divide by \(h\) and obtain \(a = q_ah + r_a\), where \(q_a \in \mathcal{P}_{w-6}\), \(r_a \in \mathcal{P}_w\) and \(r_a\) has the form \(r_a = r_a'(y)x + r_a''(y)\), with \(r_a'(y)\) quasi-homogeneous of weight \(w - 3\) and \(r_a''(y)\) quasi-homogeneous of weight \(w\).

**Case** \(w' = 0, w = 6k\): Since \(w - 3\) is odd, we deduce that \(r_a'(y) = 0\) and so \(r_a = r_a''(y) = 2\). We want to prove that if \(a \in \mathcal{P}_{6k} - \{0\}\) and \(e \notin 6\mathbb{N}\), then \((\delta^2 - ey)(a) \neq 0\). We proceed by induction on \(k\). For \(k = 1\) the result has already been proved. Assume that \((\delta^2 - ey)(a') \neq 0\) whenever \(a' \in \mathcal{P}_{6(k-1)} - \{0\}\) and take \(a \in \mathcal{P}_{6k}, a \neq 0\). We have

\[
(\delta^2 - ey)(a) = (\delta^2 - ey)(q_a) + (\delta^2 - ey)(r_a) = (\delta^2 - ey)(q_a) = (\delta^2 - ey)(q_a) \neq 0 \text{ by the induction hypothesis.}
\]

**Case** \(w' = 2, w = 6k + 2\): Since \(w - 3\) is odd, we deduce that \(r_a'(y) = 0\) and so \(r_a = r_a''(y) = 6\gamma y^k, \gamma \in \mathbb{C}\). We want to prove that if \(a \in \mathcal{P}_{6k+2} - \{0\}\) and \(e \notin 6\mathbb{N}\), then \((\delta^2 - ey)(a) \neq 0\). We proceed by induction on \(k\). For \(k = 0\) the result has already been proved. Assume that \((\delta^2 - ey)(a') \neq 0\) whenever \(a' \in \mathcal{P}_{6(k-1)+2} - \{0\}\) and take \(a \in \mathcal{P}_{6k+2}, a \neq 0\). By using the formulae [5] we find that

\[
(\delta^2 - ey)(a) \equiv \gamma((6k + 2)(6k + 3) - e)y^{3k+2} \pmod{h},
\]

and so \((\delta^2 - ey)(a) \neq 0 \text{ if } \gamma \neq 0\). If \(\gamma = 0\) then \(q_a \neq 0\) and \((\delta^2 - ey)(a) = (\delta^2 - ey)(q_a) \neq 0 \text{ by the induction hypothesis.}
\]

**Case** \(w' = 3, w = 6k + 3\): Since \(w\) is odd, we deduce that \(r_a''(y) = 0\) and so \(r_a = r_a'(y) = 18\gamma y^{k+1}, \gamma \in \mathbb{C}\). We want to prove that if \(a \in \mathcal{P}_{6k+3} - \{0\}\) and \(e \notin 6\mathbb{N}\), then \((\delta^2 - ey)(a) \neq 0\). We proceed by induction on \(k\). For \(k = 0\) the result has already been proved. Assume that \((\delta^2 - ey)(a') \neq 0\) whenever \(a' \in \mathcal{P}_{6(k-1)+3} - \{0\}\) and take \(a \in \mathcal{P}_{6k+3}, a \neq 0\). By using the formulae [5] we find that

\[
(\delta^2 - ey)(a) \equiv \gamma((6k + 3)(6k + 4) - e)y^{3k+1} x \pmod{h},
\]

and so \((\delta^2 - ey)(a) \neq 0 \text{ if } \gamma \neq 0\). If \(\gamma = 0\) then \(q_a \neq 0\) and \((\delta^2 - ey)(a) = (\delta^2 - ey)(q_a) \neq 0 \text{ by the induction hypothesis.}
\]

**Case** \(w' = 5, w = 6k + 5\): Since \(w\) is odd, we deduce that \(r_a''(y) = 0\) and so \(r_a = r_a'(y) = 24\gamma y^{k+2}, \gamma \in \mathbb{C}\). We want to prove that if \(a \in \mathcal{P}_{6k+5} - \{0\}\) and \(e \notin 6\mathbb{N}\), then \((\delta^2 - ey)(a) \neq 0\). We proceed by induction on \(k\). For \(k = 0\) the result has already been proved. Assume that \((\delta^2 - ey)(a') \neq 0\) whenever \(a' \in \mathcal{P}_{6(k-1)+5} - \{0\}\) and take \(a \in \mathcal{P}_{6k+5}, a \neq 0\). By using the formulae [5] we find that

\[
(\delta^2 - ey)(a) \equiv \gamma((6k + 5)(6k + 6) - e)y^{3k+2} x \pmod{h},
\]
and so \((\delta^2 - ey)(a) \neq 0\) if \(\gamma \neq 0\). If \(\gamma = 0\) then \(q_0 \neq 0\) and \((\delta^2 - ey)(a) = b(\delta^2 - ey)(q_0) \neq 0\) by the induction hypothesis.

The remaining cases \(w' = 1\) and \(w' = 4\) will be treated otherwise. For any weight \(w\) and any \(a \in P_w\), \(a \neq 0\), there are unique \(d \geq 0\) and unique polynomials \(a_0, \ldots, a_d\) such that

\[
a = a_d h^d + a_{d-1} h^{d-1} + \cdots + a_0, \quad a_d \neq 0, \quad a_i = a_i'(y) x + a_i''(y).
\]

Moreover, \(a_i\) is quasi-homogeneous of weight \(w - 6i\), \(a_i'(y)\) is quasi-homogeneous of weight \(w - 6i - 3\) and \(a_i''(y)\) is quasi-homogeneous of weight \(w - 6i\).

**Case** \(w' = 1, w = 6k + 1\) \((k \geq 1)\): Let \(a \in P_w\) be a non-zero element and consider the unique expression

\[
a = a_d h^d + a_{d-1} h^{d-1} + \cdots + a_0\text{ as before. Since } w - 6i \text{ is odd, we deduce that } a_i''(y) = 0 \text{ and so } a_i = a_i'(y) x = \gamma_i y^{3(k-i)-1} x, \quad \gamma_i \in \mathbb{C}. \text{ Note that in this case } d < k. \text{ Let us call } b = (\delta^2 - ey)(a) \in P_{w+2}. \text{ By using the formulae } [5] \text{ we find that the unique expression } [6] \text{ for } b \text{ is given by}
\]

\[
b = (\delta^2 - ey) \left( \sum_{i=0}^{d} \gamma_i y^{3(k-i)-1} x^i \right) = \sum_{i=0}^{d} \gamma_i h^i (\delta^2 - ey)(y^{3(k-i)-1} x^i) = \sum_{i=0}^{d} \gamma_i h^i \left[ 4(3(k-i) - 1)(3(k-i) - 2) y^{3(k-i)-3} x + \left( (6(k-i) + 1)(6(k-i) + 2) - e \right) y^{3(k-i)} x \right] = \left[ \gamma_d 4(3(k-d) - 1)(3(k-d) - 2) y^{3(k-d)-3} x \right] h^{d+1} + b_d h^d + \cdots + b_0.
\]

Since \(\gamma_d \neq 0\) \((a_d \neq 0)\), we obtain that \(b_{d+1} \neq 0\) and so \(b \neq 0\).

We conclude that \((\delta^2 - ey) : P_{6k+1} \to P_{6k+3}\) is injective for all \(k \geq 1\) and all \(e \in \mathbb{C}\).

**Case** \(w' = 4, w = 6k + 4\) \((k \geq 0)\): Let \(a \in P_w\) be a non-zero element and consider the unique expression \([6] a = a_d h^d + a_{d-1} h^{d-1} + \cdots + a_0\). Since \(w - 6i\) is even, we deduce that \(a_i''(y) = 0\) and so \(a_i = a_i'(y) = \gamma_i y^{3(k-i)+2}\), \(\gamma_i \in \mathbb{C}\).

Let us call \(b = (\delta^2 - ey)(a) \in P_{w+2}\). By using the formulae \([5]\) we find that the unique expression \([6]\) for \(b\) is given by

\[
b = (\delta^2 - ey) \left( \sum_{i=0}^{d} \gamma_i y^{3(k-i)+2} x^i \right) = \sum_{i=0}^{d} \gamma_i h^i (\delta^2 - ey)(y^{3(k-i)+2}) = \sum_{i=0}^{d} \gamma_i h^i \left[ 4(3(k-i) + 2)(3(k-i) + 1) y^{3(k-i)+1} h + (6(k-i) + 4)(6(k-i) + 5)y^{3(k-i)+3} \right] = \left[ \gamma_d 4(3(k-d) + 2)(3(k-d) + 1) y^{3(k-d)+1} \right] h^{d+1} + b_d h^d + \cdots + b_0.
\]

Since \(\gamma_d \neq 0\) \((a_d \neq 0)\), we obtain that \(b_{d+1} \neq 0\) and so \(b \neq 0\).

We conclude that \((\delta^2 - ey) : P_{6k+4} \to P_{6k+6}\) is injective for all \(k \geq 0\) and all \(e \in \mathbb{C}\). Q.E.D.

(4.17) Theorem. For \(\lambda = -2, f = 2 = (e = 2)\), we have
1. The local system $\mathcal{L}_{-2,2}$ is irreducible.

2. The image of $\mathcal{M}_{-2,2} \to \mathcal{M}_{-8,2}$ is presented as
   
   \[ \mathcal{N} = \mathcal{D}_X / \mathcal{D}_X (\chi + 2, 9y\partial_x^2 - 4\partial_y^2), \]

   and $\text{DR}(\mathcal{N}) \simeq j_* \mathcal{L}_{-2,2}$.

3. The regular holonomic $\mathcal{D}_X$-module $\mathcal{N}$ is simple and its characteristic variety is $T^*_X(X) \cup T^*_{D_{reg}}(X)$.

References


