Division theorem over the Dwork-Monsky-Washnitzer completion of polynomial rings and Weyl algebras

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Introduction

The aim of this paper is to give proofs of Weierstrass-Hironaka division theorems, in the sense of [1], over the Dwork-Monsky-Washnitzer completion of polynomial and Weyl algebras in several variables with coefficients in a complete discrete valuation ring. The case of polynomial algebras (th. 3.5) generalizes the Weierstrass division theorem stated in [14] and precises the original proof of noetherianity given in [6]. The case of Weyl algebras (th. 5.1) has been motivated by joint work with Z. Mebkhout since 1987, in order to analyze division techniques over the ring of infinite linear differential operators $D^\dagger$ over a smooth weakly formal scheme [11, 10] (compare with [12] in the complex analytic setting).

Proofs of theorems above have been obtained by the author between 1987 and 1989. A proof in the case of the full completion of Weyl algebras in one variable has been proposed independently in [7].

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§1 Rings of Dwork-Monsky-Washnitzer power series

Let \((W, m = (\pi))\) be a complete discrete valuation (CDV) ring, \(K\) its field of quotients and \(k\) its residue field. The field \(K\) becomes a complete ultrametric discrete valued field by defining \(|c| = \varepsilon^{v_m(c)}\), for every \(c \in K^*\) and \(\varepsilon \in [0,1[\). We say (cf. [13]) that a formal power series \(f(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha\) with coefficients in \(W\) (resp. in \(K\)) satisfies the Dwork-Monsky-Washnitzer condition (DMW) if there is a constant \(\lambda > 0\) such that \(v_m(f_\alpha) \geq \lambda|\alpha|\) if \(|\alpha| > 0\). The set of power series \(f(x)\) with coefficients in \(W\) (resp. in \(K\)) satisfying the DMW condition is a sub-\(W\)-algebra (resp. a sub-\(K\)-algebra) of the \(W\)-algebra (resp. the \(K\)-Banach algebra) of strictly convergent power series

\[
W(x) = \overline{W[x]} = \{ \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha \mid \lim_{|\alpha| \to \infty} v_m(f_\alpha) = +\infty \}
\]

(resp. \(K(x) = K \otimes_W W(x)\)), that will be denoted by \(W[x]^\dagger\) (resp. \(K[x]^\dagger\)). We will call the ring \(W[x]^\dagger\) the weak completion or DMW completion of \(W[x]\).

Notice that

\[
K[x]^\dagger = K \otimes_W W[x]^\dagger,
\]

and that the \(K\)-Banach algebra \(K(x)\) is the completion of the \(K\)-normed algebra \(K[x]^\dagger\).

If \(f(x) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha\) is a power series in \(K \otimes_W W[[x]]\), its \((m,\text{-adic})\) valuation is defined by

\[
v(f) := \inf \{ v_m(f_\alpha) \mid \alpha \in \mathbb{N}^n \} \in \mathbb{Z}.
\]

It is clear that \(v(fg) = v(f) + v(g)\), \(v(f + g) \geq \min\{ v(f), v(g) \}\) if \(v(f) \neq v(g)\) and that a power series \(f\) in \(K[x]^\dagger\) (resp. in \(K(x)\)) belongs to \(W[x]^\dagger\) (resp. to \(W(x)\)) if and only if \(v(f) \geq 0\).

The ring \(W[x]^\dagger\) can also be described in the following way (see [6]). Let

\[
\Phi: W[x][[Z]] \longrightarrow W[x]
\]

be the homomorphism of \(W[x]\)-algebras sending \(Z\) to \(\pi\). The image of \(\Phi\) is the \(m\)-adic completion of the polynomial ring \(W[x]\), i.e. the ring \(W[x]\). Let \(R\) be the subring of \(W[x][[Z]]\) consisting of the power series \(F = \sum_{m \geq 0} P_m(x)Z^m\) such that there is a constant \(c > 0\) with \(\deg_{x}(P_m(x)) \leq c(m+1)\) for all integers \(m \geq 0\). Then the ring \(W[x]^\dagger\) coincides with the image of \(R\) by \(\Phi\). More precisely, \(\Phi\) induces an isomorphism \(R/(\pi) \simeq W[x]^\dagger\). This (easy) result has the following consequence:

**Lemma 1.1.** Let \(f\) be a formal power series in \(W[[x]]\), and let \(\overline{f}^{(m)} \in (W/m^m)[[x]]\) denote its reduction modulo \(m^m\) for \(m \geq 1\). For \(m = 1\) we will write simply \(\overline{f} := \overline{f}^{(1)}\). The following properties are equivalent:

a) \(f\) satisfies the DMW condition.

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b) \( f^{(m)} \) is a polynomial for every \( m \geq 1 \), and there is a constant \( c > 0 \) such that \( \deg(f^{(m)}) \leq c(m + 1) \) for all \( m \geq 1 \).

§ 2 Division over polynomial rings with coefficients in a field

Let \( A \) be a ring and \( F(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n} F_\alpha x^\alpha \) a formal power series with coefficients in \( A \). The support of \( F \) is the set
\[
\mathcal{N}(F) = \{ \alpha \in \mathbb{N}^n \mid F_\alpha \neq 0 \} \subseteq \mathbb{N}^n.
\]

More generally, if \( F = (F_1, \ldots, F_m) \) is a vector of formal power series with coefficients in \( A \), there is a unique representation
\[
F = \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^n} F_{j,\alpha} x^{j,\alpha}
\]
where the \( F_{j,\alpha} \) are in \( A \) and \( x^{j,\alpha} \) denotes the vector \((0, \ldots, x^\alpha, \ldots, 0)\), the monomial being at the \( j^{th} \) place. We have \( F_j = \sum_{\alpha \in \mathbb{N}^n} F_{j,\alpha} x^\alpha \) for every \( j = 1, \ldots, m \).

The support of \( F \) is the set
\[
\mathcal{N}(F) = \{ (j, \alpha) \in \{1, \ldots, m\} \times \mathbb{N}^n \mid F_{j,\alpha} \neq 0 \} \subseteq \mathbb{N}^n \times \{1, \ldots, m\}.
\]

If \( F \) is a vector of polynomials, then \( \mathcal{N}(F) \) is a finite subset of \( \{1, \ldots, m\} \times \mathbb{N}^n \).

As usual, if \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) are two elements in \( \mathbb{N}^n \), we will consider the following total order relations:

Lexicographical order: \( \alpha <_{\text{lex}} \beta \) if there is an index \( j \) between 1 and \( n \) such that \( \alpha_i = \beta_i \) for \( i < j \) and \( \alpha_j < \beta_j \).

Inverse lexicographical order: \( \alpha <_{\text{invlex}} \beta \) if there is an index \( j \) between 1 and \( n \) such that \( \alpha_i = \beta_i \) for \( i > j \) and \( \alpha_j < \beta_j \).

Diagonal order: \( \alpha <_{d} \beta \) if either \( |\alpha| < |\beta| \) or \( |\alpha| = |\beta| \) and \( \alpha <_{\text{lex}} \beta \).

\( \Lambda \)-Order: Given a linear form \( \Lambda : \mathbb{R}^n \to \mathbb{R} \) with non negative coefficients, \( \alpha <_{\Lambda} \beta \) if either \( \Lambda(\alpha) < \Lambda(\beta) \) or \( \Lambda(\alpha) = \Lambda(\beta) \) and \( \alpha <_{\text{lex}} \beta \).

If \( \Lambda(\alpha) = |\alpha| \), then \( <_{d} = <_{\Lambda} \). If the coefficients of the linear form \( \Lambda \) are linearly independent over \( \mathbb{Z} \), then \( \alpha <_{\Lambda} \beta \) if and only if \( L(\alpha) < L(\beta) \).

If \( (i, \alpha) \) and \( (j, \beta) \) are two elements in \( \{1, \ldots, m\} \times \mathbb{N}^n \), we will also consider the following total order relations:

Semi-lexicographical \( \Lambda \)-order for vectors: Given a linear form \( \Lambda : \mathbb{R}^n \to \mathbb{R} \) with non negative coefficients, \( (i, \alpha) <_{\Lambda} (j, \beta) \) if
a) either \( \Lambda(\alpha) < \Lambda(\beta) \) or

b) \( \Lambda(\alpha) = \Lambda(\beta) \) and either \( i < j \) or \( i = j \) and \( \alpha <_{\text{lex}} \beta \).

When \( \Lambda(\alpha) = |\alpha| \) we will write \( <_d \) instead of \( <_\Lambda \).

\( \tilde{\Lambda} \)-order for vectors: Given a linear form \( \tilde{\Lambda} : \mathbb{R}^{1+n} \to \mathbb{R} \) with non-negative coefficients linearly independent over \( \mathbb{Z} \), \( (i, \alpha) \prec (j, \beta) \) if \( \tilde{\Lambda}(i-1, \alpha) < \tilde{\Lambda}(j-1, \beta) \).

Denote by \( \prec \) one of the above total order relations over \( \{1, \ldots, m\} \times \mathbb{N}_n \) or, more generally, a monomial order in \( \{1, \ldots, m\} \times \mathbb{N}_n \) (cf. [2]).

Suppose that the ring \( A \) is a field \( L \) and take a non-zero vector \( F \in L[x]^m \).

The exponent of \( F \) is defined by

\[
\exp(F) := \sup_{\prec} (\mathbb{N}(F)).
\]

If \( Q \) is a non-zero polynomial in \( L[x] \) then \( \exp(QF) = \exp(Q) + \exp(F) \), and if \( G \in L[x]^m \) is another non-zero vector with \( \exp(F) < \exp(G) \) then \( \exp(F + G) = \exp(G) \).

Recall the division theorem over \( L[x]^m \) (cf. [2]).

**Theorem 2.1.–** Let \( F_1, \ldots, F_p \in L[x]^m \) be non-zero vectors. Denote by

\[
\Delta_1 := \mathbb{N}_n + \exp(F_1)
\]

\[
\Delta_i := (\mathbb{N}_n + \exp(F_i)) \setminus \bigcup_{j=1}^{i-1} \Delta_j, \quad i = 2, \ldots, p
\]

\[
\Delta := \mathbb{N}_n \times \{1, \ldots, m\} \setminus \bigcup_{i=1}^p \Delta_i = \mathbb{N}_n \times \{1, \ldots, m\} \setminus \bigcup_{i=1}^p (\mathbb{N}_n + \exp(F_i)).
\]

Then, for every \( G \in L[x]^m \) there is a unique family of polynomials \( Q_1, \ldots, Q_p \in L[x] \) and a unique vector \( R \in L[x]^m \) such that

1. \( G = \sum_{i=1}^p Q_i F_i + R \),

2. \( \mathbb{N}(Q_i) + \exp(F_i) \subseteq \Delta_i \) for all \( i = 1, \ldots, p \),

3. \( \mathbb{N}(R) \subseteq \Delta \).

Moreover, if \( \prec \preceq <_d \) we also have \( \deg(Q_i) \leq \deg(G) - \deg(F_i) \) and \( \deg(R) \leq \deg(G) \).
§3 Division over strictly convergent or DMW power series rings

In the situation of §1, denote by \( R \) one of the rings \( W[x] \) or \( W\langle x \rangle \) and \( \prec \) a monomial order in \( \{1, \ldots, m\} \times \mathbb{N}^n \). Let \( f(x) = \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^n} f_{j,\alpha} x^{j,\alpha} \) be a non-zero vector in \((K \otimes_{W} R)^m\). We define the exponent of \( f \) as

\[
\exp(f) := \max\{ (j, \alpha) \in N(f) \mid v(f_{j,\alpha}) = v(f) = \min_{1 \leq l \leq m} v(f_l) \}.
\]

It is clear that

1. If \( v(f) = 0 \), then \( \exp(f) = \exp(f) \), where \( f = (f_1, \ldots, f_m) \in k[x] \).
2. \( \exp(f) = \exp(cf) \), for all \( c \in K \).

Lemma 3.1.– Let \( q \in K \otimes_{W} R \) be a non-zero power series and \( f, g \in (K \otimes_{W} R)^m \) two non-zero vectors. We have the following properties:

1. \( \exp(qf) = \exp(q) + \exp(f) \).
2. If \( \exp(f) \neq \exp(g) \), then \( \exp(f+g) \in \{\exp(f), \exp(g)\} \).

Proof: We may assume \( v(q) = v(f) = 0 \) and \( v(g) = \nu \geq 0 \). Let us write

\[
q = q(0) + \pi(q(>0)), \quad f = f(0) + \pi(f(>0)), \quad g = \pi^\nu g(0) + \pi^{\nu+1} g(>0),
\]

where \( q(0) \in W[x] \), \( f(0), g(0) \in W[x]^m \), \( f(>0), g(>0) \in R^m \) satisfying

\[
v(q(0)) = v(f(0)) = v(g(0)) = 0, \quad v(q(>0)), v(f(>0)), v(g(>0)) \geq 0.
\]

We have obviously \( \exp(q) = \exp(q(0)), \exp(f) = \exp(f(0)), \exp(g) = \exp(g(0)) \) and \( \exp(qf) = \exp(q(0)g(0)) \). Then, for the first part, we are reduced to the case where \( q \in W[x] \) and \( f \in W[x]^m \), \( v(q) = v(f) = 0 \), which is clear because \( v(qf) = 0 \) and

\[
\exp(qf) = \exp(qf) = \exp(qf) = \exp(qf) = \exp(q) + \exp(f).
\]

Let us prove the second part. It is enough to consider the following three cases:

A) \( \exp(f) \prec \exp(g) \), \( v(f) = 0 < v(g) \)
B) \( \exp(f) \prec \exp(g) \), \( v(f) = 0 = v(g) \)


C) \( \exp(g) < \exp(f), \ v(f) = 0 < v(g) \)

In cases A) and C) we have \( v(f + g) = 0 \) and \( \exp(f + g) = \exp(f) \). In case B) we have \( v(f + g) = 0 \) and \( \exp(f + g) = \exp(g) \).

3.2 If \((\mu, i, \alpha), (\nu, j, \beta)\) are two elements of \( \mathbb{Z} \times \{1, \ldots, m\} \times \mathbb{N}^n \), consider the total order relation \((\mu, i, \alpha) <' (\nu, j, \beta)\) if either \( \nu < \mu \) or \( \mu = \nu \) and \((i, \alpha) < (j, \beta)\).

If \( f(x) = \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^n} f_{j,\alpha} x^j, \alpha \) is a non-zero vector in \((K \otimes W) R)^m\), denote by

\[
\tilde{\exp}(f) := (v(f), \exp(f)) \in \mathbb{Z} \times \{1, \ldots, m\} \times \mathbb{N}^n,
\]

\[
\tilde{N}(f) := \{(v(f_{j,\alpha}), j, \alpha) \mid (j, \alpha) \in N(f)\} \subseteq \mathbb{Z} \times \{1, \ldots, m\} \times \mathbb{N}^n.
\]

The following lemma is a reformulation of lemma 3.1.

**Lemma 3.3.—** Let \( q \in K \otimes W R \) be a non-zero power series and \( f, g \in (K \otimes W R)^m \) two non-zero vectors. We have the following properties:

1. \( \tilde{\exp}(gf) = \tilde{\exp}(g) + \tilde{\exp}(f) \).
2. If \( \tilde{\exp}(f) < \tilde{\exp}(g) \), then \( \tilde{\exp}(f + g) = \tilde{\exp}(g) \).

The following theorem generalizes the Weierstrass division theorem of [3, 5.2.1] for strictly convergent power series with coefficients in a complete ultrametric discrete valued field.

**Theorem 3.4.—** (Division over strictly convergent power series rings) Let \( f_1, \ldots, f_p \in K(x)^m \) be non-zero vectors. Denote by

\[
\Delta_1 := \mathbb{N}^n + \exp(f_1)
\]

\[
\Delta_i := (\mathbb{N}^n + \exp(f_i)) \setminus \bigcup_{j=1}^{i-1} \Delta_j, \quad i = 2, \ldots, p
\]

\[
\overline{\Delta} := \mathbb{N}^n \times \{1, \ldots, m\} \setminus \bigcup_{i=1}^{p} \Delta_i = \mathbb{N}^n \times \{1, \ldots, m\} \setminus \bigcup_{i=1}^{p} (\mathbb{N}^n + \exp(f_i)).
\]

Then, for every \( g \in K(x)^m \) there is a unique family of strictly convergent power series \( q_1, \ldots, q_p \in K(x) \) and a unique vector \( r \in K(x)^m \) such that

1. \( g = \sum_{i=1}^{p} q_i f_i + r \),
2. \( N(q_i) + \exp(f_i) \subseteq \Delta_i \) for all \( i = 1, \ldots, p \),
3. \( N(r) \subseteq \overline{\Delta} \).
Proof: The uniqueness is a standard consequence of lemma 3.1. For the existence, we may assume that \(v(f^i) = 0\) and \(v(g) \geq 0\). In this case \(f^i, g \in W(x)^m\) and we proceed by the Hensel’s trick. First, take the canonical images 
\(F^i = f^i, G = g\) in \(k[x]\) and divide them by using theorem 2.1. Second, lift the quotients and the remainder by respecting the corresponding supports. One proves easily that the iteration of this process converges and gives the desired \(q_i\) and \(r\). The details will be made precise in the proof of theorem 3.5.

The following theorem is the main result of this §. It generalizes the Weierstrass division theorem for DMW power series stated in [14].

**Theorem 3.5.**— (Division over DMW power series rings) Let us take the monomial order \(<=\). Let \(f^1, \ldots, f^p \in (K[x])^m\) be non-zero power series and keep the notations of theorem 3.4. Then, for every \(g \in (K[x])^m \subset K(x)^m\), the quotients \(q_1, \ldots, q_p\) and the remainder \(r\) given by theorem 3.4 are in fact in \(K[x]^m\) and \((K[x])^m\) respectively.

Proof: As in the proof of theorem 3.4, we may assume \(v(f^i) = 0\), \(v(g) \geq 0\). After reduction modulo the maximal ideal of \(W\) we find vectors 
\(F^i = f^i, G = g\) in \(k[x]^m\). Put \(d_i = \deg(F^i), \delta = \max\{\deg(G), d_1, \ldots, d_m\}, d = \inf\{d_1, \ldots, d_m\}\).

Since the power series \(f^j_i = \sum \alpha f_i^j x^\alpha\) and \(g_j = \sum \alpha g_j^i x^\alpha, j = 1, \ldots, m, i = 1, \ldots, p\), satisfy the DMW condition, there is a constant \(\lambda > 0\) and an integer \(n_0 \geq 0\) such that
\[v_m(f_i^j), v_m(g_j^i) \geq \lambda |\alpha|\]
for \(|\alpha| \geq n_0\). We can take \(n_0 \geq \frac{2}{\lambda} + d\).

By theorem 2.1, there exist \(Q_1^0, \ldots, Q_p^0 \in k[x], F^0 \in k[x]^m\) such that
\[Q^0 = \sum_{i=1}^p Q_i^0 F^i + F^0\]
\[\exp(F^i) + N(Q_i^0) \subseteq \Delta_i, \forall i = 1, \ldots, p\]
\[N(F^0) \subseteq \Delta\]
\[\deg(Q_i^0) \leq \deg(G) - \deg(F^i) \leq \delta - d, \quad \deg(F^0) \leq \deg(G) \leq \delta.\]

Lift the polynomials \(Q_i^0 \in k[x]\) and the vector \(F^0 \in k[x]^m\) to 
\(q_i^0 \in W[x], t^0 \in W[x]^m\)
with the same support. There is a vector \(g^1 \in (W[x])^m\) such that 
\[g^0 := g = \sum_{i=1}^m q_i^0 f^i + t^0 + \pi g^1.\]
We have obviously
\[ \exp(f^i) + \mathcal{N}(q^0_i) \subseteq \Delta_i, \forall i = 1, \ldots, p \]
\[ \mathcal{N}(x^0) \subseteq \Xi \]
\[ \deg(q^0_i) \leq \delta - d, \quad \deg(x^0) \leq \delta. \]

Let us write
\[ q^0_i = \sum_{|\beta| \leq \delta - d} q^0_i, x^0 \cdot x^0 \]
\[ = \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^\alpha} q^0_j, x^0 \cdot x^0 = \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^\alpha} g^1_j, x^0 \cdot x^0. \]

For \( \alpha \in \mathbb{N}^\alpha \), we have
\[ \pi g^1_{j, \alpha} = g_{j, \alpha} - \sum_{1 \leq \alpha \leq p, |\beta| \leq \delta - d} q^0_{j, \alpha} f^0_{j, \alpha - \beta} - r^0_{j, \alpha}. \]

If \(|\alpha| \geq n_0 + (\delta - d) > \delta\), then \( r^0_{j, \alpha} = 0 \) and
\[ 1 + v_m(g^1_{j, \alpha}) \geq \inf \{ v_m(g_{j, \alpha}), v_m(q_{\beta})^0 + v_m(f_{j, \gamma})^1 \mid 1 \leq i \leq p, |\beta| \leq \delta - d, \beta + \gamma = \alpha \} \geq \inf \{ \lambda|\alpha|, \lambda|\gamma| \text{ s.t. } |\beta| \leq \delta - d, \beta + \gamma = \alpha \} = \cdots = \lambda(|\alpha| - (\delta - d), j \}
for every \( j = 1, \ldots, m \). Therefore, \( v_m(g^1_{j, \alpha}) \geq 1 \) if \(|\alpha| \geq n_0 + (\delta - d)\), and
\[ \deg(q^0) < n_0 + (\delta - d). \]

By induction, assume that there are polynomials \( q^0_i \in W[x], 1 \leq i \leq p, 0 \leq l \leq N \) and vectors \( x^0 \in W[x]^m, 0 \leq l \leq N, q^1_i \in (W[x])^m, 0 \leq l \leq N + 1 \) such that
\[
\begin{align*}
q^0_1 &= q \\
q^1_1 &= q^0_1 f^i_1 + x^0 + \pi q^{i+1}, 0 \leq l \leq N \\
\exp(f^i) + \mathcal{N}(q^1_i) \subseteq \Delta_i, 1 \leq i \leq p, 0 \leq l \leq N \\
\mathcal{N}(x^0) &\subseteq \Xi, 0 \leq l \leq N \\
\deg(q^0_i) &\leq (\delta - d) + l(n_0 - d), 1 \leq i \leq p, 0 \leq l \leq N \\
\deg(x^0) &\leq \delta + l(n_0 - d), 0 \leq l \leq N \\
1 + v_m(g^1_{j, \alpha}) &\geq \lambda(|\alpha| - [(\delta - d) + (l - 1)(n_0 - d)])
\end{align*}
\]
for \(|\alpha| \geq \delta + l(n_0 - d), j = 1, \ldots, m \) and \( l = 1, \ldots, N + 1 \). In particular, we have \( \deg(q^0) < \delta + l(n_0 - d) \) for every \( l = 1, \ldots, N + 1 \).
By proceeding as in the first step of the induction we find polynomials
$q_1^{N+1}, \ldots, q_m^{N+1} \in W[z]$ and vectors
$r^{N+1} \in W[x]^m, q^{N+2} = \sum_{j=1}^{m} \sum_{\alpha \in N^m} q_j^{N+1} z^{j,\alpha} \in (W[z])^m$
such that
\[
q^{N+1} = \sum_{i=1}^{m} q_i^{N+1} f^i + q^{N+1} + \pi g^{N+2}
\]
\[
\exp(f^i) + \mathcal{N}(q^{N+1}) \subseteq \Delta_i, \forall i = 1, \ldots, p
\]
\[
\deg(q_i^{N+1}) \leq \delta - d + (N + 1)(n_0 - d)
\]
\[
\deg(q^{N+1}) \leq \delta + (N + 1)(n_0 - d),
\]
hence
\[
1 + v_m(g^{N+2}) \geq \lambda(|\alpha| - [(\delta - d) + (N + 1)(n_0 - d)])
\]
for $|\alpha| \geq \delta + (N + 2)(n_0 - d)$. In particular, $\deg(g^{N+2}) < \delta + (N + 2)(n_0 - d)$.

Therefore, we can find sequences \( \{q_i^l\}_{l \geq 0} \subset W[x], \{r^l\}_{l \geq 0} \subset W[x]^m, \{g^l\}_{l \geq 0} \subset (W[z])^m \)
satisfying the above properties.

For every $N \geq 0$, we have
\[
q = \sum_{l=0}^{N} \sum_{i=1}^{m} \pi^l q^l_i f^i + \sum_{l=0}^{N} \pi^l r^l + \pi^{N+1} g^{N+1}.
\]

To end the proof of the theorem we define
\[
q_i := \sum_{l=0}^{\infty} \pi^l q^l_i, \quad i = 1, \ldots, p
\]
\[
r := \sum_{l=0}^{\infty} \pi^l r^l.
\]

The inequalities (3) and (4) imply that the quotients $q_i$ and the remainder $r$ satisfy the DMW condition and, by (1) and (2),
\[
\exp(r^i) + \mathcal{N}(q_i) \subseteq \Delta_i, \quad \forall i = 1, \ldots, p,
\]
\[
\mathcal{N}(r) \subseteq \overline{\Delta}.
\]

\[\square\]

\textbf{Remark 3.6.–} In the proof of theorem 3.5 it is essential to control the degree of the quotients and the remainder in each induction step. For this, we have considered the exponent of power series and vectors with respect to the monomial order relation $\preceq_{\preceq_d}$ (see the last part of the theorem 2.1). However, theorem 3.4 remains true for all monomial order relations.
§4 Division over Weyl algebras

Definition 4.1.— (cf. [8, 1, §3]) Let $A$ be a ring. The Weyl algebra over $A$ of order $n \geq 1$, $W_n(A)$, is the associative unitary $A$-algebra generated by $x_1, \ldots, x_n, D_1, \ldots, D_n$ with relations:

1. $[x_i, a] = [D_i, a] = 0$, $\forall i = 1, \ldots, n$, $\forall a \in A$
2. $[x_i, x_j] = [D_i, D_j] = 0$, $\forall i, j = 1, \ldots, n$
3. $[D_i, x_j] = \delta_{ij}$, $\forall i, j = 1, \ldots, n$.

If $\alpha, \beta \in \mathbb{N}^n$, we have the relation

$$D^\alpha x^\beta = \sum_{\gamma \leq \alpha, \alpha - \gamma \leq \beta} \binom{\alpha}{\gamma} \frac{\beta!}{(\beta - \alpha + \gamma)!} x^{\alpha - \gamma} D^\gamma. \quad (6)$$

Hence the elements of $W_n(A)$ have a unique representation as finite sums

$$\sum_{\alpha, \beta \in \mathbb{N}^n} b_{\alpha, \beta} x^\alpha D^\beta \quad \text{(or} \quad \sum_{\alpha, \beta \in \mathbb{N}^n} b_{\alpha, \beta} x^\alpha D^\beta \text{)}$$

where the $b_{\alpha, \beta}$ belong to $A$.

The algebra $W_n(A)$ has a natural discrete increasing total filtration:

$$F^d W_n(A) = \left\{ \sum_{|\alpha| + |\beta| \leq d} a_{\alpha, \beta} x^\alpha D^\beta \mid a_{\alpha, \beta} \in A \right\} = \left\{ \sum_{|\alpha| + |\beta| \leq d} a_{\alpha, \beta} x^\alpha D^\beta \mid a_{\alpha, \beta} \in A \right\}$$

for every $d \geq 0$, and $F^d W_n(A) = 0$ for $d < 0$. If $B$ is a non-zero element of $W_n(A)$, its total degree is defined by

$$\deg(B) := \inf \{ d \geq 0 \mid B \in F^d W_n(A) \} \geq 0$$

and its symbol is defined by

$$\sigma(B) := B + F^{\deg(B)-1} W_n(A) \in \mathfrak{gr}_F W_n(A).$$

The associated graded ring of $W_n(A)$ for the filtration $F^\bullet$ is canonically isomorphic to the commutative polynomial ring

$$A[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n],$$
where \( \xi \) (resp. \( \eta \)) corresponds to \( \sigma(x_i) \) (resp. to \( \sigma(D_i) \)). It is clear that, under this isomorphism, \( \text{deg}(B) = \text{deg}(\sigma(B)) \) and, if \( C \in \mathcal{W}_n(A) \) is another non-zero element with \( \sigma(B) \sigma(C) \neq 0 \), then \( \sigma(BC) = \sigma(B) \sigma(C) \). In fact, the filtration \( F^\bullet \) satisfies

\[
[F^d \mathcal{W}_n(A), F^{d'} \mathcal{W}_n(A)] \subseteq F^{d+d'-2} \mathcal{W}_n(A)
\]

for all \( d, d' \geq 0 \).

If \( B = (B_1, \ldots, B_m) \) is a vector in \( \mathcal{W}_n(A)^m \), we have a unique representation

\[
B = \sum_{j=1}^m \sum_{\alpha, \beta \in \mathbb{N}_n^2} b_{j, \alpha, \beta} x_j^{\alpha} D_{\beta}.
\]

The support of \( B \) is the set

\[
\mathcal{N}(B) = \{(j, \alpha, \beta) \in \{1, \ldots, m\} \times \mathbb{N}_n^2 \mid b_{j, \alpha, \beta} \neq 0\} \subseteq \{1, \ldots, m\} \times \mathbb{N}_n^2.
\]

We extend the filtration \( F^\bullet \) to \( \mathcal{W}_n(A)^m \) and the notion of total degree to vectors in the standard way.

Suppose that the ring \( A \) is a field \( L \) and take a monomial order relation \( \prec \) in \( \{1, \ldots, m\} \times \mathbb{N}_n^2 \). For each non-zero vector \( B \in \mathcal{W}_n(L)^m \) we define the exponent of \( B \) by (cf. [4])

\[
\exp(B) := \exp(\sigma(B)),
\]

where \( \sigma(B) \) is taken as a vector in \( L[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n]^m \).

If \( Q \) is a non-zero element of \( \mathcal{W}_n(L) \), then \( \exp(QB) = \exp(Q) + \exp(B) \) and, if \( B' \in \mathcal{W}_n(L)^m \) is another non-zero vector with \( \exp(B) \prec \exp(B') \), then \( \exp(B + B') = \exp(B') \).

In [4, 5] we find a division theorem for vectors in \( \mathcal{W}_n(L)^m \) in the case that \( L \) is a field of characteristic zero. In fact, the proof does not depend on the characteristic of the ground field. Therefore, we have the following result (loc. cit.)

**Theorem 4.2.** Let \( L \) be a field and \( B^1, \ldots, B^p \in \mathcal{W}_n(L)^m \) non-zero vectors. Let us denote by

\[
\Delta_1 := \exp(B^1) + \mathbb{N}_n^2
\]

\[
\Delta_i := (\exp(B^1) + \mathbb{N}_n^2) \setminus \bigcup_{j=1}^{i-1} \Delta_j, \ i = 2, \ldots, p
\]

\[
\Delta := \mathbb{N}_n^2 \setminus \bigcup_{i=1}^p \Delta_i = \mathbb{N}_n^2 \setminus \bigcup_{i=1}^p (\exp(B^i) + \mathbb{N}_n^2).
\]

Then, for every \( A \in \mathcal{W}_n(L)^m \) there exists a unique family of elements \( Q_1, \ldots, Q_p \in \mathcal{W}_n(L) \) and a unique vector \( R \in \mathcal{W}_n(L)^m \) such that

\[
1. \ A = \sum_{i=1}^p Q_i B^i + R.
\]

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2. $N(Q_i) + \exp(B^i) \subseteq \Delta_i$ for all $i = 1, \ldots, p$.

3. $N(R) \subseteq \Delta$.

Moreover, if $\prec \triangleq <_d$, we also have $\deg(Q_i) \leq \deg(A) - \deg(B^i)$ and $\deg(R) \leq \deg(A)$. 

§5 Completion of Weyl Algebras over CDV rings

In the situation of §1, consider the $m$-adic topology on the Weyl algebra $\mathcal{W}_n(W)$. We have natural ring isomorphisms

$$
\mathcal{W}_n(W)/m^l\mathcal{W}_n(W) \simeq \mathcal{W}_n(W/m^l), \quad l \geq 0,
$$

and the elements of $\mathcal{W}_n(W)$ have a unique power series representation with coefficients in $W$

$$
\sum_{\alpha, \beta \in \mathbb{N}^n} b_{\alpha, \beta} x^\alpha D^\beta
$$
satisfying the convergence condition $v_m(b_{\alpha, \beta}) \to \infty$ for $|\alpha| + |\beta| \to \infty$.

Following [13, 6], we define the weak completion, or DMW completion, of $\mathcal{W}_n(W)$ as the subring $\mathcal{W}_n(W)^\dagger$ of $\mathcal{W}_n(W)$ consisting of the elements $B$ such that, for some constant $c > 0$, the reduction of $B$ modulo $m^l$ has total degree $\leq c(l + 1)$ for every $l \geq 1$. In other words, the power series representation

$$
\sum_{\alpha, \beta \in \mathbb{N}^n} b_{\alpha, \beta} x^\alpha D^\beta
$$
of the elements in $\mathcal{W}_n(W)^\dagger$ satisfies the stronger convergence condition $v_m(b_{\alpha, \beta}) \geq \lambda(|\alpha| + |\beta|)$, for a constant $\lambda > 0$ and $|\alpha| + |\beta| \gg 0$.

We will denote by

$$
\mathcal{W}_n(K) := K \otimes_W \mathcal{W}_n(W), \quad \mathcal{W}_n(K)^\dagger := K \otimes_W \mathcal{W}_n(W)^\dagger.
$$

If $\bar{b} \in \mathcal{W}_n(K)^m$ is a non-zero vector,

$$
\bar{b} = \sum_{j=1}^m \sum_{\alpha, \beta \in \mathbb{N}^n} b_{j, \alpha, \beta} x^\alpha D^\beta,
$$

we define the valuation and the support of $\bar{b}$, denoted $v(\bar{b}) \in \mathbb{Z}$ and $N(\bar{b}) \subseteq \{1, \ldots, m\} \times \mathbb{N}^n$ respectively, in the usual way. If $v(\bar{b}) = v$, we set

$$
\text{in}(\bar{b}) := \sum_{1 \leq j \leq m, \alpha, \beta \in \mathbb{N}^n, v_m(b_{j, \alpha, \beta}) = v} b_{j, \alpha, \beta} x^\alpha D^\beta \in \mathcal{W}_n(K)^m.
$$
It is clear that $\pi^{-v} \text{in}(\bar{b}) \in \mathcal{W}(W)^m$.

We define the exponent of $\bar{b}$ as

$$\exp(\bar{b}) := \exp(\text{in}(\bar{b})) \in \{1, \ldots, m\} \times \mathbb{N}^{2n},$$

where $\text{in}(\bar{b})$ is taken as a vector in $\mathcal{W}(K)^m$, i.e.

$$\exp(\bar{b}) = \exp(\text{in}(\bar{b})) = \exp(\sigma(\text{in}(\bar{b}))).$$

Notice that, if $v(\bar{b}) = 0$, then

$$\exp(\bar{b}) = \exp(\bar{b}'),$$

where $\bar{b} \in \mathcal{W}(k)^m$ is the reduction of $\bar{b}$ modulo $m$.

Following the proof of lemma 3.1, one can easily see that, if $q \in \hat{\mathcal{W}}(K)$ is a non-zero element, then $v(q\bar{b}) = v(q) + v(\bar{b})$, and if $\bar{b}' \in \mathcal{W}(L)^m$ is another non-zero vector with $\exp(\bar{b}) \neq \exp(\bar{b}')$, then $\exp(\bar{b} + \bar{b}') \in \{\exp(\bar{b}), \exp(\bar{b}')\}$.

The following theorem generalizes theorems 3.4 and 3.5 to the non-commutative case.

**Theorem 5.1.**— (Division over the rings $\hat{\mathcal{W}}(K)$ and $\hat{\mathcal{W}}(K)^\dagger$) Let $\bar{b}_1, \ldots, \bar{b}_p$ be non-zero vectors in $\hat{\mathcal{W}}(K)^m$ (resp. in $(\hat{\mathcal{W}}(K)^\dagger)^m$, and assume that $\prec = <_{\text{d}}$).

Let us write

$$\Delta_1 := \exp(\bar{b}_1) + \mathbb{N}^{2n},$$

$$\Delta_i := (\exp(\bar{b}_i) + \mathbb{N}^{2n}) \setminus \bigcup_{j=1}^{i-1} \Delta_j, i = 2, \ldots, m,$$

$$\Delta := \{1, \ldots, m\} \times \mathbb{N}^{2n} \setminus \bigcup_{i=1}^{m} \Delta_i = \{1, \ldots, m\} \times \mathbb{N}^{2n} \setminus \bigcup_{i=1}^{m} (\exp(\bar{b}_i) + \mathbb{N}^{2n}).$$

Then, for every $\bar{a} \in \hat{\mathcal{W}}(K)^m$ (resp. $\in (\hat{\mathcal{W}}(K)^\dagger)^m$) there exists a unique family of elements $q_1, \ldots, q_p \in \mathcal{W}(K)$ (resp. $\in \mathcal{W}(K)^\dagger$) and a unique vector $\bar{r} \in \mathcal{W}(K)^m$ (resp. $\in (\hat{\mathcal{W}}(K)^\dagger)^m$) such that

1. $\bar{a} = \sum_{i=1}^{p} q_i \bar{b}_i + \bar{r}$,

2. $N(q_i) + \exp(\bar{b}_i) \subseteq \Delta_i$ for all $i = 1, \ldots, p$,

3. $N(\bar{r}) \subseteq \Delta$.

Moreover, $v(q_i) \geq v(\bar{a}) - v(\bar{b}_i)$ and $v(\bar{r}) \geq v(\bar{a})$.

**Proof:** The uniqueness is a standard consequence of properties of exponents. For the existence, we will only treat the case of DMW completions.
The proof in the case of $\mathbb{W}_n(K)$ is easier, and is in fact contained in the first one.

We may assume $v(b^i) = 0$, $v(a) \geq 0$ for $i = 1, \ldots, p$. Put $a = (a_1, \ldots, a_m), b^i = (b_1^i, \ldots, b_m^i)$ and, for every $j = 1, \ldots, m, i = 1, \ldots, p$,

$$a_j = \sum_{\alpha, \beta \in \mathbb{N}_n} a_{j, \alpha, \beta} x^\alpha D^\beta$$

$$b_j^i = \sum_{\alpha, \beta \in \mathbb{N}_n} b_{j, \alpha, \beta} x^\alpha D^\beta.$$  

Take $\lambda > 0, n_0 \geq 0$ such that

$$v_m(a_{j, \alpha, \beta}), v_m(b_{j, \alpha, \beta}) \geq \lambda(|\alpha| + |\beta|)$$

if $|\alpha| + |\beta| \geq n_0$, for every $j = 1, \ldots, m, i = 1, \ldots, p$.

Put

$$d := \inf \{ \deg(b^i) \mid i = 1, \ldots, m \},$$

$$\delta := \sup \{ \deg(a), \deg(b^i) \mid i = 1, \ldots, m \}.$$

We can take

$$n_0 \geq \frac{2}{\lambda}, n_0 \geq d.$$

According to theorem 4.2, there is a unique family of elements $Q^0_1, \ldots, Q^0_p \in \mathbb{W}_n(k)$ and a unique vector $R^0 \in \mathbb{W}_n(k)^m$ such that

1. $\pi = \sum_{i=1}^p Q^0_i b^i + R^0_i,$

2. $\exp(b^i) + \mathcal{N}(Q^0_i) \subseteq \Delta_i$ for all $i = 1, \ldots, m,$

3. $\mathcal{N}(R^0) \subseteq \Delta.$

Moreover, $\deg(Q^0_i) \leq \deg(\pi) - \deg(b^i) \leq \delta - d$ and $\deg(R^0) \leq \deg(\pi) \leq \delta$.

Lift the $Q^0_i \in \mathbb{W}_n(k)$ and the $R^0 \in \mathbb{W}_n(k)^m$ to elements $q^0_i \in \mathbb{W}_n(W)$ and $r^0 \in (r^0_1, \ldots, r^0_m) \in \mathbb{W}_n(W)^m$ respecting the support. We have then

$$\exp(b^i) + \mathcal{N}(q^0_i) \subseteq \Delta_i,$$

$$\mathcal{N}(r^0) \subseteq \Delta,$$

$$\deg(q^0_i) \leq \delta - d, \deg(r^0) \leq \delta.$$

$$v(q - \sum_{i=1}^m q^0_i b^i - r^0) \geq 1.$$
Put $\mathbf{a}^0 := \mathbf{a}$; there is a vector $\mathbf{a}^1 = (a_1^1, \ldots, a_m^1) \in (W_n(K)^\dagger)^m$ such that
\[
\mathbf{a}^0 = \left[ \sum_{i=1}^p q_i^0 b_i^0 + \mathbf{z}^0 \right] = \pi \mathbf{a}^1.
\]

Let us write
\[
q_i^0 = \sum_{|\gamma| + |\tau| \leq s} a_{i,\gamma,\tau} x^\gamma D_\tau^T, \quad r_j^0 = \sum_{|\gamma| + |\tau| \leq \delta} r_{j,\gamma,\tau} x^\gamma D_\tau^T,
\]
\[
a_j^1 = \sum_{\alpha, \beta \in [N^n]} a_{j,\alpha,\beta} x^\alpha D_\beta^T, \quad j = 1, \ldots, m.
\]

By formula (6), we find, for every $j = 1, \ldots, m$ and every $\alpha, \beta \in [N^n]$, $\pi a_{j,\alpha,\beta} = a_{j,\alpha,\beta} - r_{j,\alpha,\beta} - \sum_{i=1}^p \sum_{\gamma, \mu, \nu, \sigma \in \mathbb{D}(\alpha, \beta)} q_{i,\gamma,\tau} b_{j,\mu,\nu} \left( \frac{\tau}{\sigma} \right) \frac{\mu!}{(\mu - \tau + \sigma)!}$,

where $\mathbb{D}(\alpha, \beta)$ is the set of the $(\gamma, \tau, \mu, \nu, \sigma) \in (N^n)^5$ such that $\sigma \leq \tau \leq \sigma + \mu$, $\gamma + \mu - \tau + \sigma = \alpha, \sigma + \nu = \beta$ and $|\gamma| + |\tau| \leq \delta - d$.

For $|\alpha| + |\beta| \geq n_0 + (\delta - d)$ and for $(\gamma, \tau, \mu, \nu, \sigma) \in \mathbb{D}(\alpha, \beta)$ we have then $r_{j,\alpha,\beta} = 0$,
\[
|\mu| + |\nu| = |\alpha| + |\beta| + |\tau| - 2|\sigma| - |\gamma| \geq 0,
\]
\[
|\alpha| + |\beta| + |\tau| - 2|\sigma| + |\gamma| - (\delta - d) = |\alpha| + |\beta| + |\gamma| + |\sigma| - (\delta - d) \geq 0,
\]
\[
|\alpha| + |\beta| - (\delta - d) \geq n_0,
\]
\[
v_m \left( q_{i,\gamma,\tau} b_{j,\mu,\nu} \left( \frac{\tau}{\sigma} \right) \frac{\mu!}{(\mu - \tau + \sigma)!} \right) \geq v_m (b_{j,\mu,\nu}) \geq \lambda(|\mu| + |\nu|) \geq \lambda(|\alpha| + |\beta| - (\delta - d)),
\]

and
\[
1 + v_m (a_{j,\alpha,\beta}) \geq \lambda(|\alpha| + |\beta| - (\delta - d))
\]
for all $j = 1, \ldots, m$.

Therefore,
\[
\deg(\mathbf{a}^1) < n_0 + (\delta - d).
\]

As in the proof of theorem 3.5, we construct inductively each of the sequences
\[
\{ q_i^1 \}_{1 \leq i \leq m} \subset W_n(W),
\]
\[
0 \leq i < \infty.
\]
\[ \{ r^l \}_{0 \leq l < \infty} \subset \mathcal{W}_n(W)^m, \]
\[ \{ a^l \}_{0 \leq l < \infty} \subset (\mathcal{W}_n(W)^\dagger)^m \]
satisfying that
\[ a^0 = a, \]
\[ a^l = [r^l + \sum_{i=1}^p q_i^l b^i] + \pi g^{l+1}, \forall l \geq 0, \]
\[ \exp(h^l) + \mathcal{N}(q^l) \subseteq \Delta, \text{ for all } i = 1, \ldots, m, l \geq 0, \]
\[ \mathcal{N}(h^l) \subseteq \Sigma, \text{ for all } l \geq 0, \]
\[ 1 + v_m(d_{l,\alpha,\beta}) \geq \lambda((|\alpha|+|\beta|-(\delta-d)+(l-1)(n_0-d))) \text{ for } |\alpha|+|\beta| \geq \delta+l(n_0-d) \]
and for all \( l \geq 0, \)
\[ \deg(f^l) < n_0 + l(\delta - d) \text{ for all } l \geq 0. \]
To end the proof of the existence, define
\[ q_i := \sum_{l=0}^{\infty} \pi^l r_{li}, \quad i = 1, \ldots, p, \]
\[ \varpi := \sum_{l=0}^{\infty} \pi^l r^l. \]

By the above properties, it is clear that \( q_i \in \mathcal{W}_n(W)^\dagger, \varpi \in (\mathcal{W}_n(W)^\dagger)^m \text{ and } \mathcal{N}(q_i) + \exp(h^l) \subseteq \Delta, \text{ for all } i = 1, \ldots, p, \]
\[ \mathcal{N}(\varpi) \subseteq \Sigma. \]

\[ \square \]

**Remark 5.2.**— Theorems 3.4 and 3.5 give obviously another proof (see [6]) of noetherianity for the rings \( K \langle x \rangle \) and \( K[x]^\dagger. \) In fact they can be extended to the rings \( W \langle x \rangle \) and \( W[x]^\dagger. \) For that, we must use the extended exponent \( \exp \) of vectors in \( W \langle x \rangle^m \) or \( (W[x]^\dagger)^m \) as it was defined in 3.2.

**Remark 5.3.**— As in the preceding remark, theorem 5.1 can be extended to the rings \( \mathcal{W}_n(W) \) and \( \mathcal{W}_n(W)^\dagger. \) The fact that \( \mathcal{W}_n(W)^\dagger \) is a Zariski ring implies that the extensions \( \mathcal{W}_n(W)^\dagger \subset \mathcal{W}_n(W) \) and \( \mathcal{W}_n(K)^\dagger \subset \mathcal{W}_n(K) \) are faithfully flat.

**Remark 5.4.**— A natural question is to generalize theorems 3.4, 3.5 and 5.1 to the case of arbitrary complete ultrametric valued ground fields \( K. \) In this general case, the corresponding results over the ring \( W = \{ c \in K \mid \left| c \right| \leq 1 \} \) will be true no longer, because \( W \) is noetherian if and only if \( K \) is discrete. Following the arguments in [12], we can also prove that the ring \( K \langle x \rangle \) is a faithfully flat extension of \( K[x]^\dagger. \) This will be analysed in a joint work with Z. Mebkhout [9].
References


