Orthogonal Triangles in the Plane

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Orthogonal layouts are used in many applications of Computer Science as VLSI–design, data flow diagrams and database design in Software Engineering or entity relationships diagrams [1, 2, 8]. This fact has attracted the attention of many authors and numerous results have been obtained about orthogonal drawings [3, 6, 7, 9, 10, 12, 13].

The problem of connecting without intersections pairs of points in the plane using orthogonal paths with at most one bend was studied by Raghavan et al. [12]. A pair of points to be connected is called a *wire* and this problem is known as *Single Bend Wiring*. They develop an $O(n^2)$ algorithm to determine whether or not a set of n point pairs can be wired in this manner on the plane. The paths are not allowed to cross, hence most wire sets cannot be wired in a rectilinear fashion on the plane. In this case, those authors prove that determining a maximum cardinality subset such that all their wires can be laid out in single bend fashion on the plane is an NP–hard problem. They get those results relating their *Single Bend Wiring* problem with some satisfiability problems (see [5]). In this work, we study some natural extensions of the original *Single Bend Wiring* considering triples of points instead of pairs, but relating now the original problems with known results in Computational Geometry.

More formally, the general problem studied in this work is:

Orthogonal Triples with k bends $\textsc{ot}(k)$
INSTANCE: Set of triples $\{T_1, T_2, \ldots, T_n\}$, where each T_i is a set of three points in the plane and a positive integer k.
QUESTION: Can be connected simultaneously all the points of each set T_i using an orthogonal embedding on the planar grid of K_3 with at most k bends and without intersections?

Obviously, there is no solution for $\textsc{ot}(0)$ because there is no embedding of K_3 in the planar grid without bends.

Using one bend for each orthogonal embedding of K_3 in the planar grid ($\textsc{ot}(1)$), we can only wire the triples whose points are in the vertices of a rectangle, of course, this fact can be checked in linear time. Thus, the problem is reduced to intersection of rectangles and there is an $O(n \log n)$ algorithm that solve the problem in optimal time (see [4, 11]). So, we have the following result.

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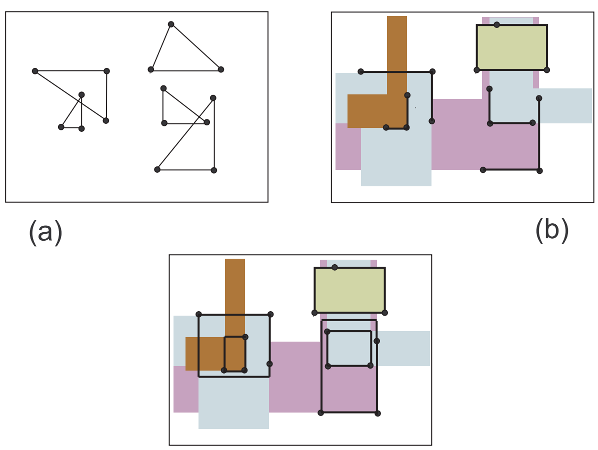
Theorem 1. OT(1) *can be solved in optimal* $O(n \log n)$ *time.*

Observe that this algorithm has better behavior that the equivalent problem for pairs of points.

If we allow two bends for orthogonal embedding of K_3 (what we have denoted as OT(2)) then there are infinitely many possibilities for each wire. In spite of this fact, a polynomial (actually, a quadratic) algorithm exists in this case. This algorithm is based again in a reduction to a problem in Computational Geometry;

Theorem 2. OT(2) *can be solved in* $O(n^2)$ *time.*

Proof: (SKETCH) In this case, for wiring the triples, they must have two of its points p_1, p_2 in the same horizontal or vertical line, say in the same horizontal. Now, depending on the position of the third point p_3, two possible cases can occur. If p_3 is not in the same vertical of p_1 or p_2, there exists only an orthogonal layout for the triple. Otherwise, if p_3 is in the same vertical of p_1 (or p_2), there are infinitely many possible layouts for the triple, all of them sharing the segments p_1p_3 and p_1p_2, and defining, in this way two orthogonal bands for each triple (in addition, one of those segment must be a side of the rectangle solution). Now, we try to find non–intersecting rectangles with the constrains given above, and we can prove that this problem can be solved in quadratic time. □



(a) (b) (c)

Figure 1: (a) An input of OT(2) problem. (b) The bands defined by this input. (c) A solution of this example.

Finally, with three or more bends in each orthogonal embedding of K_3 (OT(k), $k \geq 3$) the problem turns to be an NP–complete problem. Summarizing

Theorem 3. OT(k) *is solvable in polynomial time if and only if* $k \leq 2$. *Unless* P=NP.

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