The problem of eight points
N B Vasiliev
Quantum; Jan/Feb 1999; 9, 3; Research Library
pg. 25

AT THE BLACKBOARD I

The problem of eight points

by N. B. Vasiliev

SOMETIMES A PROBLEM WITH a simple statement has a complicated solution. On these occasions, the challenge is often to pull out a solution that is orderly, and from which one can learn. The following problem is of this type. It was suggested to students of the ninth form at the All-Soviet mathematical olympiad in Ashkhabad in 1983. It is given here with some minor modifications.

Problem 1. A circle is circumscribed around a triangle ABC. Lines AP, BP, and CP are drawn in the triangle's plane through an arbitrary point P (not on the circle), and the second points of intersection of these lines with the circle are marked. Prove that there are no more than eight points P for which the marked points do not coincide with any of the triangle's vertices A, B, and C and which are the vertices of a triangle congruent to the original triangle ABC.

This problem turned out to be rather difficult. Some students solved it by analyzing various positions of point P relative to lines AB, BC, CA, and the circle. In this article, we give a more instructive solution, motivated by the idea of "moving" figures about the plane. That is, we imagine that certain elements of our configuration rotate. When they reach a particular position, the desired figure appears.

First, we solve the following inverse problem.

Rotation and intersection of lines

Problem 2. Two (not necessarily congruent) triangles ABC and $A_1B_1C_1$ are inscribed in a circle. Triangle ABC is fixed, and triangle $A_1B_1C_1$ rotates about the center of the circle. In what positions of $A_1B_1C_1$ do the lines AA_1, BB_1, and CC_1 pass through the same point P? How many such positions are there?

The answer to the last question is as follows. Such a position is unique. That is, as triangle $A_1B_1C_1$ makes a complete rotation, lines AA_1, BB_1, and CC_1 meet at a point only once (and in a certain degenerate case, such a position does not exist).

This problem can be solved by using the method of loci.

Lemma. Let chord AB of the circle be fixed, and let the ends of chord A_1B_1 slide along the circle. Then the angle ϕ between lines AA_1 and BB_1 remains fixed, and their point M of intersection (if $\phi \neq 0$) describes a circle passing through points A and B (fig. 1).

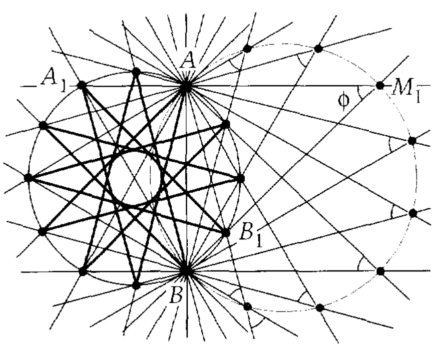

Figure 1.

If points A *and* B *are fixed and points* A_1 *and* B_1 *move uniformly with the same angular speed* ω *along the circle, lines* AA_1 *and* BB_1 *rotate uniformly with the angular speed* $\omega/2$*, and the point of their intersection,* M_1*, moves along the red circle (with the angular speed* ω*).*

Here is a way of thinking about the proof of this lemma. Assume that points A_1 and B_1 are moving uniformly with the same speed along the circle. Then lines AA_1 and BB_1 rotate uniformly with the same speed about points A and B, respectively. Therefore, the angle between them does not change (in fig. 1, for points M on one side of AB, angle AMB equals ϕ, and for points on the other side, this angle equals $\pi - \phi$). If, at the initial moment, lines AA_1 and BB_1 intersect at a point M_1, the circle circumscribed about triangle ABM_1 is the desired trajectory of point M. Point M moves uniformly along this circle (the angular speed of the rotation of the lines equals half the angular speed of the rotation of points A_1, B_1, and M along their respective circles).

A formal proof of this lemma would involve several applications of the inscribed angle theorem. The reader is invited to construct such an argument.[1]

We must note two special cases of the situation in the lemma. They will be useful in future considerations.

1. The special case $\phi = 0$ occurs when chords AB and A_1B_1 are equal, and at a certain initial moment, point A_1 coincides with B and point B_1 coincides with A. In this case, lines AA_1 and BB_1 initially coincide, and then, as chord A_1B_1 moves, they become parallel.

2. When A_1 coincides with A (or B_1 coincides with B), the line AA_1 (or BB_1) must be considered tangent to the circle (if we do not make this assumption, the two corresponding

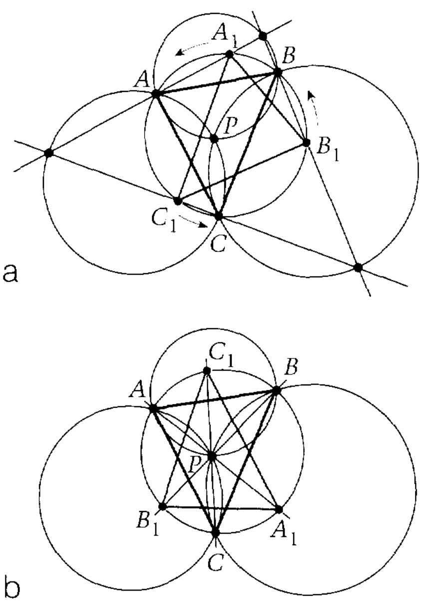

Figure 2.

Point P is at the intersection of two loci: When the line reaches the position BP, lines AA_1 and CC_1 coincide with AP and CP.

points must be excluded from the locus).

Let's return to problem 2. Using the lemma, we can construct two circles that are the loci of the points of intersection of line AA_1 with BB_1 and BB_1 with CC_1 (fig. 2). The first circle passes through points A and B, and the second passes through points B and C. Only one point can play the role of P—the point of intersection of these two circles that is different from B. We needn't construct the third locus of intersection of AA_1 and CC_1—the third circle. This third circle will pass through the same point P. A proof of this fact is left to the reader.

Thus, in general, a unique position of triangle $A_1B_1C_1$ exists that satisfies the condition required. In any case, taking into account the refinements we have made, we can say that there can be no more than one point P (the special case is shown in fig. 3). Thus, problem 2 is solved.

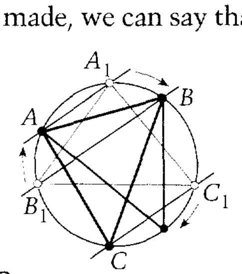

Figure 3.

A particular case $\phi = 0$: Point P does not exist (lines AA_1, BB_1, and CC_1 are parallel).

Now we can turn to problem 1. For every point P not on the circle, denote by A_1, B_1, and C_1 the second points of intersection of lines AP, BP, and CP with the circle. The statement of the problem concerns points P such that triangle $A_1B_1C_1$ is congruent to triangle ABC. At first glance, it may seem that by taking the congruent triangles in problem 2, we can find a single desired triangle $A_1B_1C_1$ that is symmetric to triangle ABC with respect to the center of the circle. However, there is a fine point in the reasoning that is more logical than geometrical, which we will discuss after a brief excursion.

Permutation of vertices and symmetries

Congruent triangles have, by definition, congruent angles and sides, and they can be superimposed. It is common practice to write the congruence of triangles so that the corresponding vertices are listed in the same order. So, for example, if $\triangle ABC \equiv \triangle DEF$, then $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, $AB = DE$, and so on.

Looking back at the statement of problem 1 with this in mind, we notice that triangle $A_1B_1C_1$ is not necessarily congruent to triangle ABC. If we take into account the correspondence of vertices, it can be congruent to any of the six triangles ABC, BCA, CAB, BAC, ACB, and CBA. And the number of variants is double this, as we will see, for purely geometrical reasons.

Triangle $A_1B_1C_1$ can be *directly* congruent to triangle ABC. That is, these triangles can be superimposed by a continuous motion on the plane (by a rotation R in our problem), or *inversely* congruent. In the latter case, in order to superimpose the triangles, we must "flip" one of them (reflect it in a line). In our problem, it suffices to reflect one of the triangles with respect to a certain line. All the triangles $A_1B_1C_1$ that are inversely congruent to triangle ABC can be obtained from each other by rotations. To differentiate between

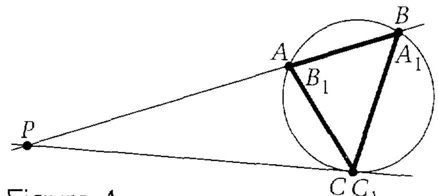

Figure 4.

A particular case: Point P appears when the triangles coincide.

these cases, we will write the letter R or S above the equality sign. For each of $2 \cdot 6 = 12$ alternatives, we can use problem 2 and construct at most one desired point P.

This reasoning can be explained as follows. We take a triangle T made of cardboard (with the same circumradius as $\triangle ABC$), place it on the plane on one of its sides or another, mark the vertices A_1, B_1, and C_1 (6 different alternatives), place its vertices on the circle, and find point P for each of the $2 \cdot 6 = 12$ alternatives by rotating this triangle.

To finish solving problem 1, we must explain why four alternatives are excluded in the case $T = \triangle ABC$. One of them ($\triangle ABC \overset{S}{=} \triangle A_1B_1C_1$) can be eliminated at once, because in this case, for each of the three sides, special case (1) of the lemma occurs (fig. 3)—lines AA_1, BB_1, and CC_1 never meet at a point as triangle $A_1B_1C_1$ rotates. Also, the condition that none of the points A_1, B_1, and C_1 coincide with the corresponding points A, B, and C eliminates the case $\triangle BAC \overset{R}{=} \triangle A_1B_1C_1$ (fig. 4) and two similar cases: $\triangle ACB \overset{R}{=} \triangle A_1B_1C_1$, $\triangle CBA \overset{R}{=} \triangle A_1B_1C_1$. In these cases, point P appears at the moment when the triangles coincide (in this case, special cases (1) and (2) of the lemma both occur (fig. 4)). Problem 1 is solved.

For those who are patient enough to finish the analyses, we recommend thinking about the following questions: (1) Is it true that in the general case, all 12 alternatives (and in the case $T = \triangle ABC$, all 8 alternatives) are realized and give (as a rule) different points P? (Try experimenting with straightedge and compass.) (2) How much does the number 12 (or the number 8 in the particular case) decrease for an isosceles or equilateral triangle T?