Thesis “Cohomology Operations: a Combinatorial Approach”

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Abstract

In this work, a purely combinatorial point of view of Steenrod and Adem cohomology operations at cocycle level is established. The goal of this study is to cover the lack of information in the literature about the underlying combinatorial structure in these cohomology operations. Working in the context of Simplicial Topology and making use of homology perturbation techniques, we design an algebraic–combinatorial machinery for the generation of cohomology operations starting from an Eilenberg–Zilber contraction. The result of this method is the normalized simplicial description of morphisms at cocycle level that determine the said operations. Finally, this explicit formulation allows us to consider this field from an algorithmic approach.

1 Extended Abstract

One of the most increasing field in pure mathematics in 20th century has been the solution of topology problems reducing them to more simplistic ways using groups.

At the beginning of 20th century, theorems in Algebraic Topology showed us almost exclusively their purely qualitative nature, that is, they asserted the (non)existence of objects almost never giving methods for determining them explicitly. This scene has been changed in these last 30 years, because a totally algorithmic framework for solving problems of invariant computation in Algebraic Topology has been established (see, for
example, [Ser87, Sch91]). In this sense, the goal of our work has been to make our contribution, giving algorithmic methods for computing elements of the cohomology of a topological space.

Roughly speaking, the philosophy of Algebraic Topology consists of associating objects that are homomorphism invariants with topological spaces.

Some of these invariants are (co)homology groups. In a certain algebraic sense, the cohomology and homology groups represent dual concepts. It is well-known that neither homology groups are able to distinguish spaces, nor cohomology groups are. But cohomology groups have an additional algebraic frame: a ring structure. This allows us to distinguish spaces whenever homology groups can not. The product in this ring is called cup product and the explicit description of its formula was given by Čech and Whitney.

More concretely, let \( X \) be a topological space and \( G \) a group. Cohomology groups of \( X \) are denoted by \( H^q(X;G) \), for all integer \( q \geq 0 \). Every \( H^q(X;G) \) is a quotient group. Its elements are class. Every element that belongs to a cohomology class constitutes a representative cocycle of this class.

In [Ste47], Steenrod introduced cohomology operations that are topological invariants finer than (co)homology groups or the cup product, moreover, these operations are a generalization of the cup product. More precisely, working with finite simplicial complexes, Steenrod introduced a family of operations, \( Sq^i : H^q(X;F_2) \rightarrow H^{q+i}(X;F_2) \), called later Steenrod squares. The description of these operations were very difficult to handled in Steenrod own words. In 1949, working with Alexander–Spanier cohomology groups, Cartan in [Car50] gave a simpler presentation of Steenrod construction.

Steenrod in [Ste52] set a family of new cohomology operations now called Steenrod reduced powers. He obtained this result with the introduction of a new definition of \( Sq^i \) making use of the Lefschetz definition of the cup product. In this definition, the higher homotopy commutativity (measured in terms of certain maps \( D_i : C^n(X) \rightarrow C^n(X) \otimes C^n(X), \ i \geq 0 \) of the cup product has an important role. Nevertheless, no general explicit formulae of \( D_i \) were given, because of Steenrod used the Acyclic Model Method [EM53a] for guaranteed the existence of these maps. Roughly speaking, we could say that the acyclic model method establishes the existence of morphisms and chain homotopies using the fact that the homology groups of same “models” are null. In the context of Simplicial Topology, this method can be considered as a constructive process, but one obtains recursive formulae for \( D_i \).

In our work, we are interested in covering this fault in the study of these operations. We give a computational tool that allows us not only to obtain an explicit description for Steenrod squares and reduced powers, but also to establish a general algebraic–combinatorial machinery that provides us a “minimal” simplicial formulation for many
cohomology operations. In like manner, this deep combinatorial analysis also gives us the possibility of designing algorithms for computing cocycles (via these formulae).

More concretely, when we began this work, we had asked for the following problem:

Does there exist any general and computationally feasible algorithm, such that the input is a polyhedron $X$, a representative cocycle $c$ and an integer $i$, and the output is a representative cocycle of the cohomology class of the Steenrod squares $Sq^i(c)$?

It is necessary to note that we design algorithms at cocycle level. In this way, when we talk about computing the explicit formula of a cohomology operation, we want to say that we pretend to obtain a combinatorial expression of the image of a representative cocycle of a cohomology class via this cohomology operation.

So, we are not worried, at first, if a cocycle obtained with a cohomology operation is representative of the null cohomology class or not. But, the step to cohomology in the finite case or locally finite case (finite in each degree) is a simple problem of Linear Algebra. In any case, our method provides us a first general computational algorithm for detecting representative cocycles of possibly non null cohomology classes. Moreover, this algorithm could be improved making use of well-known algebraic properties for these cohomology operations and homology computational techniques based on contractions.

Due to the fact that Steenrod squares can be defined using the morphisms $D_i$, our problem can be translated to that of finding an explicit formulation of such morphisms.

In order to give a solution of our problem, we move in the context of Simplicial Topology, where the basic objects are simplicial sets. A simplicial set is a graded set endowed with two kind of operators: face operators $\partial_i$ and degeneracy operators $s_j$ satisfying several commutativity relations.

Working in this field, Real in [Rea96] found the explicit formulation of $D_i$ in terms of the component morphisms $(AW, EML, SHI)$ of a contraction Eilenberg–Zilber (an special kind of homotopy equivalence) from $C_\ast(X \times X)$ onto $C_\ast(X) \otimes C_\ast(X)$. In this description, the morphisms $AW$ and $SHI$ appear. The explicit formula of this last one has been recently discovered by Sergeraert and Rubio [Rub91].

In this way, Real obtained an explicit combinatorial description for $D_i$. The problem is that $SHI$ is determined by “shuffles” (an special kind of permutation) of degeneracy operators and, consequently, the number of summands in its formula is very high. Due to this fact, the specification of $D_i$ has a very high number of summands as well.

Because of this, the idea of simplifying this formulae appears in a natural way.
This simplification or normalization is based on the fact that any composition of face
and degeneracy operators of a simplicial set $X$ can be expressed in a “canonical” way.
Working in this manner, we obtain a combinatorial expression of the morphisms $D_i :$
$C^N_*(X) \rightarrow C^N_*(X) \otimes C^N_*(X), \quad i \geq 0,$
only in terms of face operators of the simplicial
set $X$.

Afterwards, making use of an analogous scheme to that in [Rea96], we establish
formulae of Steenrod reduced powers,

$$\mathcal{P}_i^p : H^q(X; F_p) \rightarrow H^{q-i}(X; F_p), \quad p \text{ being an odd prime.}$$

These cohomology operations are generalizations of Steenrod square. The formulae we
obtain for these operations are in terms of the component morphisms of an Eilenberg–
Zilber contraction from $C^N_*(X \times \overset{p \text{ times}}{\cdots} \times X)$ onto $C^N_*(X) \otimes \overset{p \text{ times}}{\cdots} \otimes C^N_*(X)$.

In this sense, we could work with Steenrod cyclic reduced powers without effort.

Starting from this general method, we continue the work of normalization for Steen-
rod reduced powers and we present an explicit combinatorial formulation for the oper-
ations $\mathcal{P}_i^p, \quad i = 1, 2$.

Now, the following question appears:

Is it possible to express any cohomology operations via Eilenberg–Zilber
contractions?

In order to try to solve this problem, we have to introduce in the field of Homology
Perturbation Theory. In this context, we develop a technique that allows us to “see”
all the set of Steenrod cohomology operations.

In our approximation we use fiber bundles, taking as a datum the normalized cochain
complex of a classifying space of a finite cyclic group. The reason to do this approxima-
tion is to closely follow the original Steenrod and Adem work. More concretely, using
homology perturbation techniques, we rediscover the formulae of Steenrod homotopies
given previously with a more global vision. What is more, with this scheme, we study
more general cohomology operations like Adem cohomology operations of second kind
[Ade52, Ade58, Ade62].

At the same time, this algebraic–geometric machinery of generation of cohomology
operations at cocycle level, allows us to consider this field with an algorithmic vision.
In this sense, we design algorithms that compute cocycles starting from another ones
in simplicial complexes, via this explicit formulation. Moreover, we analyze a first
complexity of these algorithms in terms of the number of face operators required in the
explicit formulae.
Now we describe briefly the results in this work.

In first chapter, we give simplicial and algebraic background for the understanding of this work. In the last section of this chapter, we give several classical definitions of Steenrod cohomology operations.

**Definition 1.1** [Ste47] Let $X$ be a simplicial set. A higher diagonal approximation, is a family of graded homomorphisms

$$D_i : C^N_s(X) \rightarrow C^N_s(X) \otimes C^N_s(X)$$

of $i$ degree such that

$$D_0 = AW \Delta,$$

$$d \otimes D_{i+1} + (-1)^i D_{i+1} d = T D_i + (-1)^{i+1} D_i;$$

where $d$ and $d_\otimes$ are the differential on $C^N_s(X)$ and $C^N_s(X) \otimes C^N_s(X)$, respectively; and $T : C^N_s(X) \otimes C^N_s(X) \rightarrow C^N_s(X) \otimes C^N_s(X)$ is such that $T(a \otimes b) = b \otimes a$.

**Definition 1.2** [Ste47] Let $G$ a group and $X$ a simplicial set, the $i$–cup product of both cochain $c \in C^p(X; G)$ and $c' \in C^q(X; G)$, is a new cochain denoted by $c \cup_i c'$ such that

$$c \cup_i c'(x) = \mu(c \otimes c')D_i(x) \in C^{p+q-i}(X; G)$$

where $x \in C^N_{p+q-i}(X)$ and $\mu$ being the operation on $G$.

**Definition 1.3** [Ste47] Let $F_2$ be the ground ring and $X$ a simplicial set. Steenrod squares

$$Sq^k : H^p(X; F_2) \rightarrow H^{p+k}(X; F_2)$$

are defined at cocycle level by

$$Sq^k(c) = \begin{cases} c \cup_{p-k} c & \text{if } k \leq p, \\ 0 & \text{if } k > p, \end{cases}$$

where $c \in Z^p(X; F_2)$ (that is, $c$ is a $p$–cocycle).

In second chapter, we explain the work made by Real in [Rea96] for $i$–cup products.

**Theorem 1.4** [Rea96] Let $X$ be a simplicial set and let $(AW, EML, SHI)$ be the component morphisms of an Eilenberg–Zilber contraction from $C^N_s(X \times X)$ onto
Let \( \text{Theorem 1.5} \) at cocycle level. Afterwards, in fourth chapter we refine this formulation to obtain the higher diagonal approximation \( D_n : C^n_\ast(X) \to C^n_\ast(X) \otimes C^n_\ast(X) \), \( n \geq 0 \) are defined by the formula

\[
D_n = AW(t \operatorname{SHI})^n \Delta
\]

where \( t : C^n_\ast(X \times X) \to C^n_\ast(X \times X) \) is such that \( t((x, y)) = (y, x) \); and \( \Delta : C^n_\ast(X) \to C^n_\ast(X \times X) \) is such that \( \Delta(x) = (x, x) \).

In second section of the said chapter, we derive a normalized combinatorial expression for the morphisms \( D_i \) and consequently, for the cup product and Steenrod squares at cocycle level. Afterwards, in fourth chapter we refine this formulation to obtain the following “minimal” combinatorial expression for the cup product.

**Theorem 1.5** Let \( F_2 \) be the ground ring, \( X \) a simplicial set, \( G \) a commutative group and \( n \) a non-negative integer. If \( c \) is a \( p \)-cocycle and \( c' \) is a \( q \)-cocycle then

- If \( n \) is even,
  \[
  c \sim_n c' = \sum_{0 \leq t_0 = S(0) < t_1 < \ldots < t_n \leq m} (-1)^{t_0 + A \left( \frac{n-2}{2} \right) + \varphi_n} 
  \mu(c(\partial_{t_0+1} \cdots \partial_{t_{i_1-1}} \partial_{t_{i_1+1}} \cdots \partial_{t_{i_n-1}} \partial_{t_{i_n+1}} \cdots \partial_{t_m} x) 
  \otimes c'(\partial_0 \cdots \partial_{t_0-1} \partial_{t_1+1} \cdots \partial_{t_{n-2}} \partial_{t_{n-1}+1} \cdots \partial_{t_{n-1}} x)).
  \]

- If \( n \) is odd,
  \[
  c \sim_n c' = \sum_{0 \leq t_0 = S(0) < t_1 < \ldots < t_n \leq m} (-1)^{t_0 + A \left( \frac{n-1}{2} \right) + \varphi_n} 
  \mu(c(\partial_{t_0+1} \cdots \partial_{t_{i_1-1}} \partial_{t_{i_1+1}} \cdots \partial_{t_{n-2}} \partial_{t_{n-1}+1} \cdots \partial_{t_{n-1}} x) 
  \otimes c'(\partial_0 \cdots \partial_{t_0-1} \partial_{t_1+1} \cdots \partial_{t_{n-1}} \partial_{t_{n+1}} \cdots \partial_{t_m} x)).
  \]

where

- \( m = p + q - n \);
- \( x \in C^m_\ast(X) \);
- we identify \( i_{n+1} \) with \( m \) if it appears;
- \( \varphi_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ i_n & \text{otherwise;} \end{cases} \)
- \( A(\ell, \bar{i}) = \sum_{0 \leq j \leq \ell} (i_{2j} + i_{2j+1})(i_{2j+1} + i_{2j+2}) 
  + \sum_{0 \leq j \leq \ell-1} (i_{2j} + i_{2j+1} + 1) \left( \sum_{j+1 \leq \ell} (i_{2\ell+1} + i_{2\ell+2} + 1) \right), \)

for all \( \ell \geq 0 \) and \( \bar{i} = (i_0, i_1, \ldots, i_{2\ell+2}) \).
In third section of the second chapter, we generalized the above–mentioned work to Steenrod reduced powers. 

**Definition 1.6** \[\text{[Ste52]}\]

Let \( X \) be a simplicial set and \( p \) an odd prime. There exists a sequence \( \{D_i\}_{i \geq 0} \) from \( C^*(X) \) onto \( C^*_*(X) \otimes p \times \cdots \otimes C^*_*(X) \) (denoted by \( C^*_*(X)^{\otimes p} \)) verifying that 

\[
d \otimes D_i + (-1)^{i+1} D_i d = \alpha_i D_{i-1};
\]

where \( d \) and \( d \otimes \) denote, the differential on \( C^*_*(X) \) and \( C^*_*(X)^{\otimes p} \), respectively; and \( \alpha_i : C^*_*(X)^{\otimes p} \to C^*_*(X)^{\otimes p}, i \geq 0 \), is defined by:

\[
\alpha_i = \begin{cases} 
T - 1 & \text{if } i \text{ is odd,} \\
1 + T + \cdots + T^{p-1} & \text{if } i \text{ is even;}
\end{cases}
\]

where \( T : C^*_*(X)^{\otimes p} \to C^*_*(X)^{\otimes p} \) is such that \( T(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_2 \otimes \cdots \otimes x_n \otimes x_1 \).

The reduced power cohomology operations, \( \mathcal{P}^p_i : H1(X; F_p) \to H^{pq-i}(X; F_p), i \geq 0 \), is defined at cocyclic level by 

\[
\mathcal{P}^p_i(c) = \mu(c \otimes \cdots \otimes c) D_i, 
\]

c being a \( q \)-cocycle and \( \mu \) the product on \( F_p \).

**Theorem 1.7** Let \( p \) be an odd prime, \( F_p \) the ground ring and \( X \) a simplicial set. Then, the sequence 

\[
\{D_n : C^*_*(X) \to C^*_*(X)^{\otimes p}\}_{n \geq 0} \text{ such that } D_n = f \gamma_n \phi \gamma_{n-1} \cdots \phi \gamma_1 \phi \Delta,
\]

is a higher diagonal approximation, where

- \((f, g, \phi)\) is an Eilenberg–Zilber contraction from \( C^*_*(X^p) \) onto \( C^*_*(X)^{\otimes p} \),
- \( \Delta : C^*_*(X) \to C^*_*(X^p) \) is such that \( t(x) = (x, p \times \cdots, x) \),
- \( \gamma_i : C^*_*(X^p) \to C^*_*(X)^{\otimes p}, i \geq 0 \) is defined by 

\[
\gamma_i = \begin{cases} 
 t & \text{if } i \text{ is even,} \\
 t + t^2 + \cdots + t^{p-1} & \text{otherwise;}
\end{cases}
\]

where \( t : C^*_*(X)^{\otimes p} \to C^*_*(X^p) \) is such that \( t((x_1, x_2, \ldots, x_n)) = (x_2, \ldots, x_n, x_1) \).

In fourth section of the second chapter, we make a combinatorial analysis of the first and second Steenrod reduced powers at chain level and we give their explicit formulae in terms of face operators. In fourth chapter, we refine the above–mentioned formula of the first power in order to obtain the following “minimal” combinatorial expression.
Theorem 1.8 Let $p$ be an odd prime, $F_p$ the ground ring and $X$ a simplicial set. If $c$ is a $q$-cocycle and $x \in C^N_{pq-1}(X)$, then $\mathcal{P}^p_1(c) \in Z^{w-1}(X; F_p)$ is described by

$$\mathcal{P}^p_1(c)(x) = \sum_{q \leq j \leq q \leq (j+1)q-3 \leq (p+1)q-1} (-1)^{(i+1)(q+1)+1} \mu(c(\partial_{i+1} \cdots \partial_{pq-1} x) \otimes c(\partial_0 \cdots \partial_{2q+1} \cdots \partial_{pq-1} x) \cdots \otimes c(\partial_0 \cdots \partial_{(j-2)q-1} \partial_{(j-1)q+1} \cdots \partial_{pq-1} x) \otimes c(\partial_0 \cdots \partial_{(j-1)q-1} \partial_{(j-1)q+1} \cdots \partial_{pq-1} x) \otimes c(\partial_0 \cdots \partial_{(j+1)q-2} \partial_{(j+2)q} \cdots \partial_{pq-1} x) \cdots \otimes c(\partial_0 \cdots \partial_{(p-2)q-2} \partial_{(p-1)q-1} \cdots \partial_{pq-1} x) \otimes c(\partial_0 \cdots \partial_{(p-1)q-2} \partial_{pq-1} x) \otimes c(\partial_0 \cdots \partial_{(p-1)q-1} \partial_{pq-1} x)) ;$$

$\mu$ being the product on $F_p$.

The simplicial formula for $\mathcal{P}^p_2$ in terms of face operators is the following.

Theorem 1.9 Let $p$ be an odd prime, $F_p$ the ground ring and $X$ a simplicial set. If $c$ is a $q$-cocycle and $x \in C^N_{pq-2}(X)$ then $\mathcal{P}^p_2(c) \in Z^{w-2}(X; F_p)$ is described by

$$\mathcal{P}^p_2(x) = \sum_{1 \leq k \leq n} f_n t^k \phi_{(1,n)} t \phi_{(1,n)} \Delta(x) + \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq n-k} (1^{\otimes n-k-j} \otimes T^{k+1})(f_{n-j} t^k \phi_{(1,n-j)} t \phi_{(1,n-j)} \otimes 1^{\otimes j}) f_{(j,n)} \cdots f_{(1,n)} \Delta(x)$$

$$+ \sum_{1 \leq k \leq n} \sum_{1 \leq \ell \leq n-k} (1^{\otimes n-k-\ell+1} \otimes T^{k})(f_{n-\ell} t^k \phi_{(1,n-\ell)} \otimes 1^{\otimes \ell}) f_{(\ell,n)} \cdots f_{(1,n)} t \phi_{(1,n)} \Delta(x)$$

$$+ \sum_{1 \leq k \leq n} \sum_{1 \leq \ell \leq n-k} \sum_{1 \leq j \leq \ell-1} (1^{\otimes n-k-\ell+1} \otimes T^{k})(1^{\otimes n-j} \otimes T)(f_{n-\ell} t^k \phi_{(1,n-\ell)} \otimes 1^{\otimes \ell}) f_{(\ell,n-j)} \cdots f_{(1,n-j)} t \phi_{(1,n-j)} \otimes 1^{\otimes j}) f_{(j,n)} \cdots f_{(1,n)} \Delta(x) ;$$

where

- $n = p - 1,$
- $m = pq - 2,$

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\( f_{(r,u)} = AW \otimes 1^{\otimes r-1} : C^*_e(X^{\times u-r+2}) \otimes C^*_e(X)^{\otimes r-1} \to C^*_e(X^{\times u-r+1}) \otimes C^*_e(X)^{\otimes r}, \)

for all \( 1 \leq r \leq u, \)

\[ f_{u^k}\phi_{(1,u)}((x_1, \ldots, x_{u+1})_m) \]

\[
= \sum_{I_{(k,0,u,m)}} (-1)^{i_{u-k}+(i_u-k+i_u)(i_u+m)} \\
\quad \partial_{i_{0+1}} \cdots \partial_{i_{m}} x_{k+1} \\
\quad \otimes \partial_0 \cdots \partial_{i_{0-1}} \partial_{i_{1+1}} \cdots \partial_{i_{m}} x_{k+2} \\
\quad \cdots \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-k-2}} \partial_{i_{u-k-1}} \cdots \partial_{i_{m}} x_u \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-k-1}} \partial_{i_{u-k}} \cdots \partial_{i_{m}} x_{u+1} \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-1}} \partial_{i_{u}} \cdots \partial_{i_{m}} x_1 \\
\quad \cdots \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-1}} \partial_{i_{u}} \cdots \partial_{i_{m}} x_k,
\]

for all \( 1 \leq k \leq u \) and \( I_{(k,0,u,m)} = \{(i_0, \ldots, i_u) : 0 \leq i_0 \leq i_u \leq m, i_{u-r-s} < i_{u-s}\}, \)

\[ f_{(2,u)}f_{(1,u)}(\phi_{(1,u)}((x_1, \ldots, x_{u+1})_m) \]

\[
= \sum_{I_{(1,0,2,m)}} (-1)^{i_1+(i_1+i_2)(i_2+m)} \\
\quad (\partial_{i_{0+1}} \cdots \partial_{i_{m}} x_2 \\
\quad \cdots \\
\quad \partial_{i_{0+1}} \cdots \partial_{i_{m}} x_u) \\
\quad \otimes \partial_0 \cdots \partial_{i_{0-1}} \partial_{i_{1+1}} \cdots \partial_{i_{2-1}} x_{u+1} \\
\quad \otimes \partial_0 \cdots \partial_{i_{1-1}} \partial_{i_{2+1}} \cdots \partial_{i_{m}} x_1,
\]

where \( I_{(1,0,2,m)} = \{(i_0, \ldots, i_u) : 0 \leq i_0 \leq i_1 < i_2 \leq i_u \leq m\}, \)

\[ f_{u^k}\phi_{(1,u)}(\phi_{(1,u)}((x_1, \ldots, x_{u+1}) \otimes x_{u+1}) \]

\[
= \sum (-1)^{i_{u-k}+(i_u+w+1)(i_u+w+1)+i_{u-k-1}+(i_u-k+1+i_u+1)(i_u+1+m)+m} \\
\quad \partial_{i_{0+1}} \cdots \partial_{i_{m}} x_{k+2} \\
\quad \otimes \partial_0 \cdots \partial_{i_{0-1}} \partial_{i_{1+1}} \cdots \partial_{i_{m}} x_{k+3} \\
\quad \cdots \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-k-3}} \partial_{i_{u-k-2}} \cdots \partial_{i_{m}} x_{u} \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-k-2}} \partial_{i_{u-k-1}} \cdots \partial_{i_{m}} x_{u+1} \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-k-1}} \partial_{i_{u-k}} \cdots \partial_{i_{m}} x_{u+1} \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-1}} \partial_{i_{u}} \cdots \partial_{i_{m}} x_2 \\
\quad \cdots \\
\quad \otimes \partial_0 \cdots \partial_{i_{u-1}} \partial_{i_{u}} \cdots \partial_{i_{m}} x_{k+1},
\]
Theorem 1.10 Let us consider the $\text{DGA–algebra} \ Z_p[\mathbb{Z}_p]$. Let $X$ be a simplicial set and let us consider two actions $\nu$ and $\nu'$ defined in the following manner. Let $x \in C^N_\ast(X^{\times p})$, $y \in C^N_\ast(X)^{\otimes p}$ and $\sum \lambda_g g \in Z_p[\mathbb{Z}_p]$; then

$$\nu(x, g) = \nu(x, \sum \lambda_g g) = \sum \lambda_g t^g(x)$$

and

$$\nu'(y, g) = \nu'(y, \sum \lambda_g g) = \sum \lambda_g t^g(y).$$

Let us consider the universal twisting cochain $\theta : \hat{B}(Z_p[Z_p]) \to Z_p[Z_p]$ and the Eilenberg–Zilber contraction

$$c_{EZ} = (AW, EML, SHI) : C^N_\ast(X^{\times p}) \Rightarrow C^N_\ast(X)^{\otimes p}.$$

Using that $EML \otimes 1_{\text{id}(Z_p[Z_p])}$ commutes with $\nu$ and $\nu'$ and homological perturbation techniques, we have a contraction

$$(c_{EZ} \otimes 1)_{\theta \cap} = ((AW \otimes 1)_{\theta \cap}, EML \otimes 1, (SHI \otimes 1)_{\theta \cap})$$

from $C^N_\ast(X^{\times p}) \otimes_\theta \hat{B}(Z_p[Z_p])$ onto $C^N_\ast(X)^{\otimes p} \otimes_\theta \hat{B}(Z_p[Z_p])$.

Let $x \in C^N_\ast(X)$ and $e_i = [\tilde{\gamma}_1| \cdots |\tilde{\gamma}_i] \in \hat{B}(Z_p[Z_p])$ such that for each $1 \leq j \leq i$, $\tilde{\gamma}_j : Z_p[Z_p] \to Z_p[Z_p]$ is $1 - 0$ if $j$ is odd and $0 + 1 + \cdots + p - 1$ otherwise.

With the augmentation filtration in $\hat{B}(Z_p[Z_p])$, denoted by $F_j(Z_p)$, we have a filtration of $C^N_\ast(X)^{\otimes p} \otimes_\theta \hat{B}(Z_p[Z_p])$ such that

$$F_j(C^N_\ast(X)^{\otimes p} \otimes_\theta \hat{B}(Z_p[Z_p])) = C^N_\ast(X)^{\otimes p} \otimes_\theta F_j(Z_p), \quad \text{for all } j \in \mathbb{Z}.$$

In this situation, the morphism $(AW \otimes 1)_{\theta \cap}$ “generates” a higher diagonal approximation $\{D_i\}_{i \geq 0}$, where, up to sign,

$$D_i(x) = (AW \otimes 1)_{\theta \cap}[F_0(\Delta(x) \otimes e_i),$$

$[F_0$ being the projection in $C^N_\ast(X)^{\otimes p} \otimes_\theta Z_p]$. 

In third chapter, first of all, we study the problem of cohomology operations using homology perturbation techniques and afterwards, we give a constructive method for obtaining Steenrod reduced powers.
Following the same scheme, we obtain an explicit formulae for the morphism $E_3$, in terms of component morphisms of Eilenberg–Zilber contractions and $k$–cup products. Adem used this morphism in order to define a cohomology operation of second kind. In a second step, we express the following formula of $E_3$ in terms of face operators using the normalization techniques already studied.

**Theorem 1.11** Let $X$ be a simplicial set and $c$ a 2–cocycle. Then

$$E_3(c^4)(x) = \mu(c(\partial_1 \partial_2 \partial_3 x) \otimes c(\partial_3 \partial_4 \partial_5 x) \otimes c(\partial_0 \partial_1 \partial_2 x) \otimes c(\partial_0 \partial_1 \partial_4 x)$$

$$+ c(\partial_1 \partial_2 \partial_3 x) \otimes c(\partial_0 \partial_1 \partial_2 x) \otimes c(\partial_3 \partial_4 \partial_5 x)$$

$$+ c(\partial_2 \partial_3 \partial_4 x) \otimes c(\partial_0 \partial_1 \partial_2 x) \otimes c(\partial_0 \partial_4 \partial_5 x)$$

$$+ c(\partial_3 \partial_4 \partial_5 x) \otimes c(\partial_0 \partial_1 \partial_3 x) \otimes c(\partial_0 \partial_1 \partial_5 x)$$

$$+ c(\partial_3 \partial_4 \partial_5 x) \otimes c(\partial_0 \partial_1 \partial_4 x) \otimes c(\partial_0 \partial_1 \partial_2 x) \right).$$

where $x \in C_5^0(X)$.

Let us remark that with this result and the explicit formulae for $k$–cup products, one can obtain a normalized simplicial formula for the cohomology operation of second kind

$$\Phi : \text{Ker}(Sq^2 : H^2(X; \mathbb{Z}) \to H^4(X; \mathbb{F}_2)) \longrightarrow \text{Coker}(Sq^2 : H^3(X; \mathbb{Z}) \to H^5(X; \mathbb{F}_2)).$$

In fourth chapter, we study the complexity of the combinatorial formulae we give in second chapter.

**Theorem 1.12** The number of summands taking part in the formula of $c \sim_n c'$ from Theorem 1.5 is

$$\left( q - \left\lfloor \frac{n+1}{2} \right\rfloor \right) \left( p - \left\lfloor \frac{n}{2} \right\rfloor \right) \left( \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor \right) .$$

**Lemma 1.13** The number of summands taking part in the formula of $P^p_n(c)$ from Theorem 1.8 is

$$p(p - 1)q((p - 1)q - 1).$$

In this last chapter, we also design algorithms for computing cocycles in simplicial complexes.
Algorithm for computing $n$–cup products

**Input:** A commutative group $G$, a simplicial complex $P$, $c \in C^p(P; G)$ and $c' \in C^q(P; G)$.

Construct the set $C$ of all the $p$–simplices such that $x \in C$ if and only if $c(x) \neq 0$.

Construct the set $C'$ such that $y \in C'$ if and only if $c'(y) \neq 0$.

$D := \{} \}.$

for $x \in C, \ y \in C'$ do

$z := x \cup y = \langle v_0, \ldots, v_m \rangle$,

if $x \cap y = \langle v_{i_0}, \ldots, v_{i_m} \rangle$

is a $n$–simplex with $n = p + q - m$,

$i_0 = S(0)$ and $x = \bigcup_j \text{par } z^j$ then

$D := D \cup \{(x, y)\}.$

dendif;
endfor;

$\text{cup} := 0.$

for $(x, y) \in D_z$ do

$\text{cup} := \text{cup} + (-1)^{i_0 + A(\left \lfloor \frac{n-1}{2} \right \rfloor) + \partial_n \mu(c(x) \otimes c(y))} z.$
endfor;

**Output:** a formal sum $\text{cup} = \sum \lambda_j z_j$ such that

if $\lambda z$ ($\lambda \in G$ and $z$ is a $m$–simplex) is a summand of $\text{cup}$ then

$c \overset{n}{\sim} c'(z) = \lambda$, where $n = p + q - m$.

Otherwise, $c \overset{n}{\sim} c'(z) = 0$.

Some of the results we present in this work have already been presented in national congress [GR98c, GR98d, GR99f], international congress [AAGR98, AGJR98, GR98a, GR98e, GR99c, GR99d, GR99e], national papers [GR98b] and international papers [GR99a, GR99b].

**References**


