SOME INDEFINITE NONLINEAR EIGENVALUE PROBLEMS

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Dedicated to Prof. Jean Mawhin for his first 60 years of Nonlinear Analysis

In this work we study the structure of the set of positive solutions of a nonlinear eigenvalue problem with a weight changing sign. Specifically, the reaction term arises from a population dynamic model. We use mainly bifurcation methods to obtain our results.

1. Introduction

The aim of this work is to study some nonlinear indefinite eigenvalue problems of the form

$$\begin{cases} -\Delta u = \lambda m(x)f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a regular boundary $\partial \Omega$, $m \in C(\overline{\Omega})$ changes sign, $f$ is a regular function and $\lambda$ plays the role of real parameter. We focus our attention on the case $f(0) = 0$ and $\lambda > 0$; similar results can be obtained for negative values of $\lambda$.

Depending of the shape of $f$, Eq. (1) models different situations: population dynamics, population genetics, combustion theory,... see [10].

In the linear case, i.e., $f(u) = u$, (1) is the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda m(x)u \text{ in } \Omega, \\ u = 0 \text{ on } \partial \Omega, \end{cases}$$

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It is well known (see for instance [19] and [23]) that there exist two values of \( \lambda, \lambda_-(m) < 0 < \lambda_+(m) \), called principal eigenvalues because they have associated positive eigenfunctions. In the present work, given \( q \in L^\infty(\Omega) \) we denote by \( \sigma_1^\Omega[-\Delta + q] \) (we delete the superscript \( \Omega \) when no confusion arises) the principal eigenvalue of the problem

\[
-\Delta u + q(x)u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]

When in (1) the weight does not appear, i.e., \( m \equiv 1 \), the nonlinear problem

\[
-\Delta u = \lambda f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \quad (3)
\]

has been extensively studied. Classical references are [2] and [21], but many others can be given where, as well as existence results, uniqueness ones are shown: [4], [14], [26], [20], [22] and references therein.

Much less is known for problem (1). In [19], assuming for example that \( f'(0) > 0 \), the authors showed that there exists an unbounded continuum of positive solutions bifurcating from the trivial solution at \( \lambda = \lambda_+(m)/f'(0) \).

In [8] the authors assumed that \( f : I \mapsto \mathbb{R}_+ \), \( I \subset \mathbb{R} \), and \( f'' < 0 \) and showed that every positive solution of (1) is stable. If, moreover, \( I = [0, 1] \), \( f(1) = 0 \) and \( f'(0) > 0 \) they proved that there exists a positive solution if, and only if, \( \lambda > \lambda_+(m)/f'(0) \), and in this case the solution is unique. Similar result was shown in [13], although the authors’ motivation was to study the problem in the whole space. Very recently, in [9] the authors analyze the particular cases \( f(u) = g_i(u), i = 1, 2 \) with

\[
g_1(u) = u - u^2, \quad g_2(u) = u + u^2. \quad (4)
\]

Observe that the result of [8] can only be applied to \( g_1 \). In [9], without the assumption that \( f \) takes only values in \([0, 1]\), the main result of [8] was improved showing (by variational method) that, assuming some restriction in the space dimension, there exists positive solution if \( \lambda \in (0, \lambda_+(m)) \). For the case, \( f = g_2 \), they also proved the existence of positive solution for \( \lambda \in (0, \lambda_+(m)) \) and that there does not exist positive solution at \( \lambda = \lambda_+(m) \). In [16] these results have been again completed. We prove for \( f = g_1 \) that there exist at least two positive solutions in \( \lambda \in (\lambda_+(m), \infty) \), one of them linearly asymptotically stable and that for \( f = g_2 \) there exists positive solution if, and only if, \( \lambda \in (0, \lambda_+(m)) \).

In this work, we are going to analyze the following nonlinearities

\[
f_1(u) = u - u^2 - K \frac{u}{1+u}, \quad f_2(u) = u + u^2 - K \frac{u}{1+u}. \quad (5)
\]
where $K \in \mathbb{R}$. Observe that the functions in (4) are included in (5). These last nonlinearities arise in population dynamics. Indeed, when $K = 0$, $f_1$ is the classical logistic reaction term and for $K \neq 0$ the predation one $Ku/(1 + u)$ is called the Holling-Tanner term, see for example [7] for an ecological interpretation.

In order to state our main results we need some notations. Specifically, assume that $M_\pm := \{ x \in \Omega : m^\pm > 0 \}$ are open and regular sets, where $m^\pm$ represent the positive and negative part of $m$ respectively; and suppose that $m^\pm(x) \approx \text{dist}(x, \partial M_\pm)$ for $x$ close to $\partial M_\pm$ and some $\gamma_\pm \geq 0$. The following condition will provide us with a priori bounds of the solutions

$$2 < \min \left\{ \frac{N + 1 + \gamma_\pm}{N - 1}, \frac{N + 2}{N - 2} \right\}. \quad (6)$$

Finally, we define for $K \neq 1$ the values

$$\lambda_+ := \frac{\lambda_+(m)}{1 - K}, \quad \lambda_- := \frac{\lambda_-(m)}{1 - K},$$

and $\Pi : \mathbb{R} \times C(\Omega) \rightarrow \mathbb{R}$ the projection map onto $\mathbb{R}$, i.e. $\Pi(\mu, u) = \mu$. The main results are:

**Theorem 1.1.** Assume that $K \neq 1$ and (6).

1. There exists an unbounded continuum $C$ of positive solutions of (1) bifurcating from the trivial solution at $\lambda = \lambda_+$ if $K < 1$ and $\lambda = \lambda_-$ if $K > 1$.
2. The bifurcation is supercritical for $f = f_1$ and for $f = f_2$ and $K < -1$ or $K > 1$ and subcritical for $f = f_2$ and $K \in [-1, 1)$.
3. If $f = f_1$ and $K < 1$ (resp. $f = f_2$ and $K > 1$), then $\Pi(C) = (\lambda_+, \infty)$ (resp. $(\lambda_-, \infty)$). Moreover, if $(\lambda_\ast, u_\lambda) \in C$, then $u_\lambda$ is linearly asymptotically and such that $u_\lambda \leq \sqrt{1 - K}$ (resp. $\sqrt{K - 1}$). Furthermore, there exists another positive solution $v_\lambda$ for all $\lambda > 0$.
4. If $f = f_1$ and $K > 1$ (resp. $f = f_2$ and $K < -1$) then $\Pi(C) = (0, \lambda_\ast]$ for $\lambda_\ast > \lambda_-$ (resp. $\lambda_+)$). Moreover, there exist $\lambda_0$ and $\lambda^\ast$ with $\lambda_0 < \lambda^\ast$ such that for $\lambda \geq \lambda^\ast$ the problem (1) does not admit positive solutions and it possesses at least two positive solutions for $\lambda \in (\lambda_-, \lambda_0)$ (resp. $\lambda_+, \lambda_0$).
5. If $f = f_2$ and $K \in [-1, 1)$ there exists positive solution for $\lambda \in (0, \lambda_\ast)$ and (1) does not admit positive solutions for $\lambda \geq \lambda^\ast$.
(6) In any case, if there exists a solution $v_\lambda$ for $\lambda > 0$, then
$$\lim_{\lambda \to 0} \|v_\lambda\|_\infty = +\infty.$$ 

**Theorem 1.2.** Assume $K = 1$ and (6). Then there exists at least a solution $u_\lambda$ for $\lambda > 0$ and $\lim_{\lambda \to 0} \|u_\lambda\|_\infty = +\infty$.

**Remark 1.1.**

1. The existence of $C$ is true without assuming (6). In the cases (4) and (5) of Theorem 1.1, $C$ could “go to infinity” in a value $\lambda^0$.
2. In the particular case $f = f_2$ and $K = 0$, in [16] it was proved using a Picone inequality that (1) possesses a positive solution if, and only if, $\lambda \in (0, \lambda_+)$. 

In Figs. 1 and 2 we have summarized these results (the case $f = f_2$ and $K = 1$ is similar to $f = f_1$ and $K = 1$).

![Bifurcation diagrams](image)

Figure 1. Bifurcation diagrams for $f = f_1$: a) $K < 1$; b) $K = 1$; c) $K > 1$.

The rest of the paper is organized as follows: Secs. 2 and 3 are devoted to prove Theorems 1.1 and 1.2, respectively.

**2. Proof of Theorem 1.1**

**2.1. Local bifurcation**

In this subsection we show the direction of bifurcation from the trivial solution for both cases $f_1$ and $f_2$. For that, we write the nonlinearity of the following manner

$$f(u) = u \mp u^2 - K \frac{u}{1 + u} = u(1 - K) + u^2(\frac{K}{1 + u} \mp 1).$$
Figure 2. Bifurcation diagrams for $f = f_2$: a) $K < -1$; b) $K \in [-1, 1)$; c) $K > 1$.

It is clear that to study (1) is equivalent to find zeros of $L(\lambda)u - N(\lambda, u) = 0$, where

$\begin{align*}
L(\lambda)u &:= u - \lambda(-\Delta)^{-1}m(x)(1 - K)u, \\
N(\lambda, u) &:= \lambda(-\Delta)^{-1}m(x)u^2\left(\frac{K}{1 + u} \mp 1\right).
\end{align*}$

We can prove that

$N(L(\lambda_+)) = \text{Span } < \varphi^+ >$ and $\frac{d}{d\lambda}L(\lambda_+)\varphi^+ \notin R(L(\lambda_+)) \quad (7)$

where, given any linear continuous operator $L$, $N[L]$ and $R[L]$ stand for the null space and the range of $L$, respectively, and

$-\Delta \varphi^+ = \lambda_+(m)m(x)\varphi^+ \quad \text{in } \Omega, \quad \varphi^+ = 0 \quad \text{on } \partial\Omega. \quad (8)$

The first equality of (7) is trivial, for the second expression we need the following result.

**Lemma 2.1.** For any $p \geq 2$ we have that

$\int_{\Omega} m(x)(\varphi^+)^p > 0.$

**Proof:** Multiplying (8) by $(\varphi^+)^{p-1}$ we get

$\lambda_+(m)\int_{\Omega} m(x)(\varphi^+)^p = \int_{\Omega} (-\Delta \varphi^+)(\varphi^+)^{p-1} = (p-1)\int_{\Omega} |\nabla \varphi^+|^2(\varphi^+)^{p-2} > 0.$

Now, we show (7). Assume that there exists $u$ such that

$\frac{d}{d\lambda}L(\lambda_+)\varphi^+ = -(-\Delta)^{-1}m(x)(1 - K)\varphi^+ = u - (-\Delta)^{-1}m(x)\lambda_+(1 - K)u,$
then

\[ (-\Delta - \lambda_+(m(x))u = -(1 - K)m(x)\varphi^+, \]

and so, multiplying by \(\varphi^+\) we get a contradiction using Lemma 2.1.

Now, we can apply the Crandall-Rabinowitz Theorem [15] and conclude that there exists \(\delta > 0\) such that in a neighborhood of \((\lambda_+, 0)\) the nontrivial solutions of (1) are of the form

\[ u(s) = s\varphi^+ + s^2\varphi^3 + s^3\varphi_3 + o(s^3), \]

\[ \lambda(s) = \lambda_+ + s\lambda_1 + s^2\lambda_2 + o(s^2). \]

Introducing these terms in (1), using (8) and a Taylor expression of the function \(1/(1 + u(s))\), we get

\[ (-\Delta - \lambda_+(m(x))\varphi_2 = \lambda_+(x)(\varphi^+)^2(K \mp 1) + \lambda_1 m(x)(1 - K)\varphi^+, \]

and so,

\[ \lambda_1 = -\frac{\lambda_+(K \mp 1)}{1 - K} \int_{\Omega} \frac{m(x)(\varphi^+)^3}{\int_{\Omega} m(x)(\varphi^+)^2}. \] (9)

Observe that in the particular case \(f = f_2\) and \(K = -1\), \(\lambda_1 = 0\), and so we have to calculate \(\lambda_2\). It can be proved that

\[ \lambda_2 = -\frac{\lambda_+(\mp 1)}{2} \int_{\Omega} \frac{m(x)(\varphi^+)^4}{m(x)(\varphi^+)^2}. \] (10)

From (9) and (10), we conclude the paragraph (2) of Theorem 1.1. Analogously it can be treated the case \(\lambda_-\).

2.2. Non-existence results

Lemma 2.2. Assume \(f = f_1\) and \(K > 1\) or \(f = f_2\) and \(K < 1\). Then, there exists \(\lambda^* > 0\) such that for \(\lambda \geq \lambda^*\) (1) does not have positive solutions.

Proof: Assume \(f = f_1\) and \(K > 1\). Firstly observe that

\[ h(x) := x(\frac{K}{1 + x} - 1) \leq (\sqrt{K} - 1)^2, \quad \forall x \geq 0. \] (11)
Let \( u \) be a positive solution of (1). Then, using the monotony of the principal eigenvalue with respect to the domain and (11) we get

\[
0 = \sigma_1[-\Delta - \lambda m(x)(1 - K) - \lambda m(x)u\left(\frac{K}{1+u} - 1\right)] < \\
< \sigma_1^M[-\Delta - \lambda m(x)((1 - K) + (\sqrt{K} - 1)^2)] = \\
= \sigma_1^M[-\Delta - \lambda m(x)2(1 - \sqrt{K})],
\]

which is an absurdum for \( \lambda \) large.

Now, assume \( f = f_2 \) and \( K < 1 \). In this case,

\[
x\left(\frac{K}{1+x} + 1\right) \geq 0, \quad \text{if} \quad K \geq -1, \quad \forall x \geq 0,
\]

\[
x\left(\frac{K}{1+x} + 1\right) \geq - (\sqrt{-K} - 1)^2, \quad \text{if} \quad K < -1, \quad \forall x \geq 0.
\]

So, if \(-1 \leq K < 1\) we have

\[
0 = \sigma_1[-\Delta - \lambda m(x)(1 - K) - \lambda m(x)u\left(\frac{K}{1+u} + 1\right)] < \sigma_1^M[-\Delta - \lambda m(x)(1 - K)];
\]

on the other hand, for \( K < -1 \),

\[
0 = \sigma_1[-\Delta - \lambda m(x)(1 - K) - \lambda m(x)u\left(\frac{K}{1+u} + 1\right)] < \sigma_1^M[-\Delta - \lambda m(x)2\sqrt{-K}],
\]

in both cases a contradiction for large \( \lambda \).

\[ \diamond \]

2.3. Multiplicity results

To obtain multiplicity results, we include (1) in the more general equation

\[
\left\{ \begin{array}{l}
-\Delta u = \mu m(x)(1 - K)u + \lambda m(x)g(u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{array} \right.
\]

(12)

where \( g \) satisfies

\[(H_g)\quad g(0) = g'(0) = 0, \quad g''(u) < 0, \quad \lim_{s \to +\infty} \frac{g(s)}{s^2} = \beta < 0.\]

Problem (12) has attracted a great deal of attention during last years (see for example [1], [3], [5], [6], [18] and [24]) when \( m \equiv 1 \) in the first term on the right-hand side of (12) and in [11], [12] and [13] with the right-hand side of the form \( \mu h(x)u + g(x)u^p \) and restrictive conditions on \( h \) and \( g \) which are not satisfied in our case. In [16] was proved (see Fig. 3):

**Proposition 2.1.** Assume that \( g \) satisfies \((H_g), \ (6), \ K \neq 1 \) and fix \( \lambda > 0 \). Denote by

\[
\Lambda_+ := \lambda_+(m(x)(1 - K)), \quad \Lambda_- := \lambda_-(m(x)(1 - K)).
\]
Then, (12) possesses a positive solution if $\mu > \Lambda_-$. Moreover, from the trivial solution $u = 0$ emanate two unbounded in $\mathbb{R} \times C(\overline{\Omega})$ continua of positive solutions $\mathcal{C}_+ := \{ (\mu, u_\mu) \}$ and $\mathcal{C}_- := \{ (\mu, w_\mu) \}$ at $\mu = \Lambda_+$ and $\mu = \Lambda_-$, respectively. Both continua bifurcate to the right and $\Pi(\mathcal{C}_-) \supset (\Lambda_-, +\infty)$, $\Pi(\mathcal{C}_+) = (\Lambda_+, +\infty)$. Finally, for $\mu > \Lambda_+$, $u_\mu$ is linearly asymptotically stable and $u_\mu \neq w_\mu$.

**Remark 2.1.** Observe that for $K < 1$,

$$\Lambda_+ = \lambda_+ \quad \text{and} \quad \Lambda_- = \lambda_-,$$

and for $K > 1$,

$$\Lambda_+ = \lambda_- \quad \text{and} \quad \Lambda_- = \lambda_+.$$

Indeed, for example for $K > 1$, it follows that

$$\Lambda_+ = \lambda_+(m(x)(1 - K)) = \frac{\lambda_+(-m(x))}{K - 1} = \frac{-\lambda_-(m(x))}{K - 1} = \frac{\lambda_-(m(x))}{1 - K} = \lambda_-.$$

![Figure 3. Bifurcation diagram for (12) and $K < 1$.](image)

**2.4. Proof of Theorem 1.1:**

Before proving the result, we generalize a well-known result for $m \equiv 1$. The proof is coming from [8].

**Lemma 2.3.** Assume that $f$ is a regular function and $f(0) = 0$. Let $u_0$ be a positive solution of (1) such that $f(u_0) > 0$, it holds:
If \( f''(u_0) < 0 \), then \( u_0 \) is linearly asymptotically stable.

If \( f''(u_0) > 0 \), then \( u_0 \) is unstable.

**Proof:** We have to calculate the sign of the eigenvalue \( \sigma_1[-\Delta - \lambda m(x)f'(u_0)] \). Take \( \psi := f(u_0) > 0 \), then

\[
(-\Delta - \lambda m(x)f'(u_0))\psi = -f''(u_0)|\nabla u_0|^2.
\]

So, if \( f \) is concave (resp. convex) the function \( \psi \) is a supersolution (resp. subsolution) of \(-\Delta - \lambda m(x)f'(u_0)\), and then (see [23]) \( \sigma_1[-\Delta - \lambda m(x)f'(u_0)] > 0 \) (resp. < 0).

The following result is proved in Theorem 3.4 of [3] and provides us with a priori bounds for the positive solutions of (1).

**Lemma 2.4.** Assume (6). If \((\lambda, u)\) is a positive solution of (1) and \( \lambda \in J \), where \( J \) is a compact subset such that \( J \subset (0, \infty) \), then there exists a positive constant \( C \) (independent from \( \lambda \)) such that

\[
\|u\|_{\infty} \leq C.
\]

Finally, the following result is proved in [17].

**Lemma 2.5.** Assume that \( \Sigma \subset I \times C^2_0(\Omega) \), \( I \subset \mathbb{R} \) an interval, is a connected set of positive solutions of (1). Consider \( \overline{\pi} : I \rightarrow C^2_0(\Omega) \) a continuous map of supersolution for each \( \lambda \in I \), but not a solution. If \( u_0 < \overline{\pi}(\lambda_0) \) for some \((\lambda_0, u_0) \in \Sigma\), then \( u < \overline{\pi}(\lambda) \) for all \((\lambda, u) \in \Sigma\).

We are ready to prove the result. By subsec. 2.1 we know that there exists bifurcation from the trivial solution at \( \lambda = \lambda_+ \) or \( \lambda = \lambda_- \) when \( K < 1 \) or \( K > 1 \), respectively. Moreover, we can apply Theorem 6.4.3 of [25], and conclude that from \( \lambda = \lambda_+ \) or \( \lambda = \lambda_- \) bifurcates an unbounded continuum \( \mathcal{C} \) of positive solutions of (1). We would like to remark that the a detailed proof that \( \mathcal{C} \) is unbounded and it does not satisfy the other alternatives of the above mentioned result will be presented elsewhere.

Now assume \( f = f_1 \) and \( K < 1 \). It is clear that

\[
\overline{\pi} := \sqrt{1 - K}
\]

is a supersolution of (1). So, we can apply Lemma 2.5 (taking \( \lambda_0 = \lambda_+ \)) and conclude that

\[
\text{for all } (\lambda, u_\lambda) \in \mathcal{C}, \text{ we have that } u_\lambda < \sqrt{1 - K}.
\]

Moreover, \( f_1(u_\lambda) > 0 \) and \( f''_1(u_\lambda) < 0 \), and so by Lemma 2.3 we get that \( u_\lambda \) is linearly asymptotically stable.
Now, we are going to apply Proposition 2.1. Recall that in this case $\Lambda_+ = \lambda_+$ and $\Lambda_- = \lambda_-$. Taking as $g(u) = u^2 \left( \frac{K}{1 + u} - 1 \right)$, we obtain a positive solution for $\mu = \lambda$ and $\lambda \in (0, \lambda_+]$ and at least two positive solutions for $\lambda > \lambda_+$. Similarly, it can be considered the case $f = f_2$ and $K > 1$. Indeed, we only have to write

$$
\mu(-m(x))(K - 1)u + \lambda(-m(x))u^2(-K/(1 + u) - 1).
$$

Observe that $g(u) = u^2(-K/(1 + u) - 1)$ satisfies $(H_g)$ for $K > -1$, and so, Proposition 2.1 is true for $\Lambda_+ = \lambda_+(-m(x)(K - 1))$, and $\Lambda_- = \lambda_-(-m(x)(K - 1))$. And, since $K > 1$ it follows by Remark 2.1 that $\Lambda_+ = \lambda_-$. The paragraphs (4) and (5) follow easily from the existence of $C$ and Lemmas 2.2 and 2.4.

In order to prove paragraph (6), assume that there exist a sequence $(\lambda_n, u_n)_{n \in \mathbb{N}}$ of positive solution with $\lambda_n \to 0$ and $\|u_n\|_\infty \leq C$ for some $C > 0$. Since there does not exist positive solution of (1) for $\lambda = 0$, we obtain that $\|u_n\|_\infty \to 0$. We claim that this is impossible. Indeed, we define

$$
w_n = \frac{u_n}{\|u_n\|_\infty},
$$

then $w_n$ is uniformly bounded and, by passing to a suitable sequence again denoted by $w_n, w_n \to w^*$ as $n \to \infty$ for some $w^* \in C(\overline{\Omega})$ with $\|w^*\|_\infty = 1$. But,

$$
-\Delta w_n = \lambda_n m(x) \frac{f(u_n)}{\|u_n\|_\infty},
$$

and so $-\Delta w^* = 0$, which is an absurd. This concludes the proof.

3. The particular case $K = 1$

In this case, the bifurcation from the trivial solution disappears. Consider

$$
\begin{cases}
-\Delta u = \mu u + \lambda m(x)g(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\quad (14)
$$

where

$$
g(u) = u^2 \left( \frac{1}{1 + u} - 1 \right) \quad \text{or} \quad g(u) = u^2 \left( \frac{1}{1 + u} + 1 \right).
$$
Proposition 3.1. There exists a positive solution of (14) for \( \mu = 0 \).
In particular, for all \( \lambda > 0 \) there exists a positive solution of (1).

Proof: It easy to prove that this problem is in the setting of some works, see for example [3] and references therein, and then there exists an unbounded continuum \( \mathcal{S} \) of positive solutions of (14) bifurcating from \( \mu = \sigma_1[\Delta] \) and it satisfies that \( \Pi(\mathcal{S}) \supset (-\infty, \sigma_1[\Delta]) \) (see Theorem 7.1 in [3]). This concludes the proof.

References
14. P. Clément, G. Sweers, Existence and multiplicity results for a semilinear...