A symbiotic self-cross diffusion model

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Abstract

In this work we show existence and non-existence results of coexistence states for a Lotka-Volterra symbiotic model with self and cross-diffusion in one species. We study the behavior of the set of positive solutions when the cross-diffusion or the self-diffusion parameter is large.

Key Words. Lotka-Volterra, Cross-diffusion, Self-diffusion, Symbiosis, A priori estimates, Coexistence states

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1 Introduction.

In this paper we study positive solutions of the problem

\[
\begin{aligned}
-\Delta u &= u(\lambda - u + bv) \quad \text{in } \Omega, \\
-\Delta [(1 + \alpha v + \beta u)v] &= v(\mu - v + cu) \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N, N \geq 1 \), is a bounded and smooth domain. Throughout this work we assume that the parameters \( \lambda, \mu \) belong to \( \mathbb{R} \) and \( b, c, \alpha, \beta > 0 \). This system models two cooperating species, \( u \) and \( v \) denote their population densities, with self-diffusion and cross-diffusion in \( v \). Here, \( \lambda \) and \( \mu \) are the intrinsic growth rates, \( b \) and \( c \) are the coefficients of cooperation or symbiosis, and \( \alpha \) and \( \beta \) the self-diffusion and cross-diffusion interferences, respectively. Thus \( \alpha \) is related to inner disturbances of \( v \) and \( \beta \) measures the pressure of \( u \) into \( v \).

System (1.1) has been extensively studied in the competition model (\( b < 0 \) and \( c < 0 \)) and prey-predator case (\( bc < 0 \)), see for instance [19] and references therein. The symbiotic model (\( b > 0 \) and \( c > 0 \)) has received less attention, and to our knowledge only Pao in [15] has analyzed this case. In [2] the authors have concentrated in the case when \( \beta > 0 \) and \( \alpha = 0 \), not covered in [15]. Here we are able to further study system (1.1) by allowing \( \alpha > 0 \) and determining the asymptotic behavior of the solutions when either \( \alpha \to \infty \) or \( \beta \to \infty \).

When \( \beta = \alpha = 0 \), problem (1.1) is reduced to the classical Lotka-Volterra symbiotic model with linear diffusion which has been studied in [3], [5], [13], and references therein. In [15] the author proved that a sub-supersolution method works for (1.1), and this method was used in [10] to study a symbiotic model with the non-linear diffusion function

\[
(1 + \frac{1}{1 + \beta u + \alpha v})v
\]

instead of \((1 + \beta u + \alpha v)v\). We show the main existence result proved in [15] for (1.1).

Denote by \( \lambda_1 \) the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition and \( \varphi_1 \) its corresponding positive eigenfunction with \( \| \varphi_1 \|_{L^\infty(\Omega)} = 1 \). Then, in [15] the author proved the existence of positive solution if

\[
b(c + \frac{\beta}{\alpha}) < 1,
\]
and \( \lambda \) and \( \mu \) verify
\[
\lambda > \lambda_1, \quad \mu(1 - b\left( \frac{\beta}{\alpha} + c \right) - \beta b(\lambda_1 + \frac{1}{\alpha})) > \lambda_1(1 - b(c + \frac{\beta}{\alpha})) + \beta(\lambda_1 + \frac{1}{\alpha}).
\] (1.4)

The first aim of this paper is to improve this condition. Indeed, we prove existence of solutions by means of index theory with a priori estimates derived from the maximum principle and the blow-up argument. In order to state our results, we introduce some notations.

We denote by \((\theta_{\lambda}, 0)\) and \((0, \theta_{\mu, \alpha})\) the semi-trivial solutions of (1.1). Notice that when \( v = 0 \), we have \( u = \theta_{\lambda} \) which is the unique solution of the logistic equation
\[
\begin{align*}
-\Delta u &= u(\lambda - u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (1.5)
for \( \lambda > \lambda_1 \). When \( u = 0 \) we have \( v = \theta_{\mu, \alpha} \) which is the solution of
\[
\begin{align*}
-\Delta[(1 + \alpha v)v] &= v(\mu - v) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (1.6)
for \( \mu > \lambda_1 \) and every \( \alpha > 0 \). Observe that in this case, by Proposition 2.4 and (2.24), there exists a unique positive solution of (1.6).

Therefore the \((\lambda, \mu)\)-regions in the plane \( \mathbb{R}^2 \) where \((\theta_{\lambda}, 0)\) and \((0, \theta_{\mu, \alpha})\) are solutions are given respectively by \( \lambda > \lambda_1 \) and \( \mu > \lambda_1 \).

The following functions are important in the determination of coexistence regions. By a coexistence state we mean a classical solution \((u, v)\) of (1.1) where \( u > 0 \) and \( v > 0 \) in \( \Omega \). Define
\[
F(\alpha, \mu) := \begin{cases} 
\lambda_1[1; -b\theta_{\mu, \alpha}] & \text{for } \mu > \lambda_1, \\
\lambda_1 & \text{for } \mu \leq \lambda_1,
\end{cases}
\] (1.7)
and
\[
G(\beta, \lambda) := \begin{cases} 
\lambda_1[1 + \beta \theta_{\lambda}; -c\theta_{\lambda}] & \text{for } \lambda > \lambda_1, \\
\lambda_1 & \text{for } \lambda \leq \lambda_1.
\end{cases}
\] (1.8)

Here \( \lambda_1[A; B] \) stands for the principal eigenvalue of the problem
\[
-\Delta[A(x)\phi] + B(x)\phi = \lambda\phi \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial \Omega.
\]
The main properties of these functions are shown in Section 2.

We state now the main result concerning existence and nonexistence of solutions.

**Theorem 1.1.** (I) If \( \mu \leq \lambda_1 \) and \( \beta \lambda_1 \geq c \), (1.1) does not have coexistence states.

(II) If \( \mu, \lambda \leq \lambda_1 \) and \( b(c - \beta \lambda_1) < 1 + \alpha \lambda_1 \), (1.1) does not have coexistence states.

(III) If \( bc < 1 \), then there exists at least one coexistence state of (1.1) if \( (\lambda, \mu) \) verifies the following condition

\[
\lambda > F(\alpha, \mu) \quad \text{and} \quad \mu > G(\beta, \lambda). 
\]

(IV) There exists \( \beta_0 > 0 \) such that for all \( \beta > \beta_0 \) problem (1.1) possesses at least one coexistence state if \( (\lambda, \mu) \) verifies (1.9).

(V) There exists \( \alpha_0 > 0 \) such that for all \( \alpha > \alpha_0 \) problem (1.1) possesses at least one coexistence state if \( (\lambda, \mu) \) verifies (1.9).

![Figure 1: Coexistence regions](image-url)

Figure 1: Coexistence regions: a) \( \alpha = \beta = 0 \); b) \( \alpha > 0, 0 < \beta < c/\lambda_1 \); c) \( \alpha > 0, \beta = c/\lambda_1 \) and d) \( \alpha > 0, \beta > c/\lambda_1 \).
First, observe that this result improves the result in [15]. For example, condition (1.3) needs that \( bc < 1 \) and \( \beta/\alpha \) small, and our result works for \( bc < 1 \) and any \( \beta \) and \( \alpha \), and for any size of \( bc \) and \( \beta \) or \( \alpha \) large. Moreover, condition (1.4) implies that \( \lambda > \lambda_1 \), while our condition (1.9) is verified even for values \( \lambda \leq \lambda_1 \).

Denote by 
\[
\mathcal{R}_{\alpha,\beta} := \{ (\lambda, \mu) \in \mathbb{R}^2 : (\lambda, \mu) \text{ verifies (1.9)} \},
\]
that is, the coexistence region of (1.1). Observe that when \( \alpha = \beta = 0 \) (the linear diffusion case), see for example [3], the region \( \mathcal{R}_{00} := \{ (\lambda, \mu) \in \mathbb{R}^2 : \lambda > F(0, \mu) \text{ and } \mu > G(0, \lambda) \} \) is included in the coexistence region when \( bc < 1 \). However, when \( bc > 1 \) and assuming \( N \leq 5 \) the coexistence region is included in \( \mathbb{R}^2 \setminus \mathcal{R}_{00} \), see [3] or [13].

In the self-cross diffusion case, we have proved that \( \mathcal{R}_{\alpha,\beta} \) is included in the coexistence region when \( bc < 1 \), or when \( \alpha \) or \( \beta \) are large independently of the size of \( bc \). In Figure 1 we have drawn the regions \( \mathcal{R}_{\alpha,\beta} \) and \( \mathcal{R}_{00} \) in different cases, depending on the size of \( \beta \), see Section 6. Observe that we do not know the relative position between \( F(0, \mu) \) and \( F(\alpha, \mu) \). Nevertheless, we know that 
\[
\mathcal{R}_{\alpha,\beta} \to \mathcal{R}_{00} \quad \text{as } \alpha, \beta \to 0,
\]
see Proposition 3.1, it is clear that if \( \lambda, \mu > \lambda_1 \) then \( (\lambda, \mu) \in \mathcal{R}_{\alpha,\beta} \).

Secondly, we also study what happens if either \( \alpha \to \infty \) or \( \beta \to \infty \). Extremal situations like this have been addressed by us in [2] when \( \beta \to \infty \) and \( \alpha = 0 \). Other limiting states have been studied in [6], [7], [8] and [14]. When \( \beta \to +\infty \), the competition case has been analyzed in [7] when \( \alpha = 0 \) and in [14] for \( \alpha > 0 \). The case prey-predator and the cross-diffusion function as in (1.2) has been studied in [6] and [8]. Less studied is the case \( \alpha \to +\infty \), see [14] for the competition case and homogenous Neumann boundary conditions. In [14] the authors proved that when both self-diffusion parameters are large, then the competition problem has non-constant solution, which agrees with our result as we will see below.

We state next the result concerning the limit system when the cross-diffusion effect \( \beta \) tends to \( \infty \).

**Theorem 1.2.** (I) Fix \( (\lambda, \mu) \in \mathbb{R}^2 \) with \( \lambda > \lambda_1 \). Then, (1.1) does not have coexistence states if \( \beta > 0 \) is large.
(II) Assume now that $\lambda < \lambda_1$. Then every family of positive solutions $(u_\beta, v_\beta)$ of (1.1) verifies that $(\beta u_\beta, v_\beta) \to (z, w)$ as $\beta \to \infty$ uniformly in $\Omega$ where $(z, w)$ is a positive solution of

$$
\begin{cases}
-\Delta z = z(\lambda + bw) & \text{in } \Omega, \\
-\Delta [(1 + \omega w + z)w] = w(\mu - w) & \text{in } \Omega, \\
z = w = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1.10)

(III) System (1.10) does not possess any coexistence state if $\lambda \geq \lambda_1$ and has at least one coexistence state for $(\lambda, \mu) \in \mathbb{R}_0^+: \{ (\lambda, \mu) \in \mathbb{R}^2 : F(\mu, \alpha) < \lambda < \lambda_1 \}$.

In fact, for $\mu > \lambda_1$ fixed, an unbounded continuum $C$ in $\mathbb{R} \times (C_0^1(\Omega))^2$ bifurcates from the semi-trivial solution $(0, \theta_{\mu, \alpha})$ at $\lambda = F(\mu, \alpha)$ and a bifurcation to infinity at $\lambda = \lambda_1$ appears when the parameter $\lambda$ approaches $\lambda_1$.

We proceed with the system when the self-diffusion $\alpha$ tends to $\infty$.

**Theorem 1.3.** Assume that $bc < 1$. (I) Fix $(\lambda, \mu) \in \mathbb{R}^2$ with $\lambda < \lambda_1$. Then, (1.1) does not have coexistence states if $\alpha > 0$ is large.

(II) Assume now that $\lambda > \lambda_1$. Then every family of positive solutions $(u_\alpha, v_\alpha)$ of (1.1) verifies that $(u_\alpha, \alpha v_\alpha) \to (\theta_\lambda, z_\lambda)$ as $\alpha \to \infty$ uniformly in $\Omega$ where $z_\lambda$ is positive solution of

$$
\begin{cases}
-\Delta [(1 + \beta \theta_\lambda + z)z] = z(\mu + c\theta_\lambda) & \text{in } \Omega, \\
z = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1.12)

(III) Problem (1.12) possesses a positive solution if $\mu > G(\lambda, \beta)$.

Denote now by

$$R_{\alpha, \infty} := \{ (\lambda, \mu) : \lambda > \lambda_1, \mu > G(\lambda, \beta) \}$$

that is, the coexistence region in the case $\alpha \to \infty$. In Figure 2 we have drawn the coexistence regions in the limiting cases: $\beta \to \infty$ and $\alpha \to \infty$, respectively.

We give now some biological implications of these last results. If $\lambda > \lambda_1$, then by Theorem 1.2 (I) system (1.1) does not have coexistence states if $\beta > 0$ is large, in fact we
Figure 2: Coexistence regions: a) $\alpha > 0$ and $\beta \to \infty$; b) $\alpha \to \infty$ and $\beta > 0$.

will show that $v$ is driven to the extinction by $u$. Hence, the repulsive force (the pressure of $u$ into $v$) is stronger than the cooperation between the species. On the other hand, if $bc < 1$, $\lambda < \lambda_1$ and $\alpha$ large, then by Theorem 1.3 (I) system (1.1) does not have coexistence states, in this case, $u$ is driven to the extinction by $v$. Hence, when the growth rate of $u$ is small, then the cooperation is not able to recover the species $u$ for any value of $\mu > \lambda_1$, in contrast to the case $\alpha$ small. Indeed, for large $\alpha$ the species $v$ is small, and then it can not help to $u$, see Remark 3.2.

The outline of the paper is as follows. In Section 2 we present some preliminary results on the degenerate logistic equation and how some eigenvalues vary as the weights change. In Section 3 we perform a change of variables to treat system (1.1) in a more adequate form (3.29). We also make some remarks on the semi-trivial solutions and prove the first a priori bounds when $bc < 1$ as an application of the maximum principle. In Section 4 we show the a priori estimates using blow-up methods. Theorem 1.1 is proved in Section 5 by means of index theory. And Theorems 1.2 and 1.3 are proved in Section 6.

2 The degenerate logistic equation.

We need introduce some notations. Given $A \in C^2(\overline{\Omega}), B \in C(\overline{\Omega})$ with $A \geq A_0 > 0$, consider the following eigenvalue-problem

\[
\begin{cases}
-\Delta [A(x)\varphi] + B(x)\varphi = \lambda \varphi & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(2.13)
Observe that this problem is equivalent, under the change $A(x)\phi = \psi$, to

$$-\Delta \psi + \frac{B(x)}{A(x)} \psi = \lambda \frac{1}{A(x)} \psi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial \Omega.$$ 

**Proposition 2.1.** There exists the principal eigenvalue of (2.13), denoted by $\lambda_1[A; B]$, that is, an eigenvalue with positive eigenfunction, and the only one having an eigenfunction with definite sign. Moreover,

(I) $\lambda_1[A; B]$ in increasing in $B$.

(II) Assume $\lambda_1[A; B] \geq 0$ and $B_1/A_1 > B/A$, then $\lambda_1[A_1; B_1] > 0$.

(III) Assume $\lambda_1[A_1; B_1] \leq 0$ and $B_1/A_1 > B/A$, then $\lambda_1[A; B] < 0$.

**Proof.** The existence of $\lambda_1[A; B]$ and (I) are well-known, see for instance [11].

(II) Assume that $\lambda_1[A; B] \geq 0$ and $B_1/A_1 > B/A$. Observe that $\lambda_1[A_1; B_1]$ is the unique root of the map

$$\mu(\lambda) := \lambda_1[1; B_1A_1] - \frac{B_1}{A_1}.$$ 

We claim that

$$\lambda_1[1; \frac{B}{A}] \geq 0. \quad (2.14)$$

Assuming (2.14), $\mu(0) = \lambda_1[1; \frac{B_1}{A_1}] > \lambda_1[1; \frac{B}{A}] \geq 0$. The conclusion follows because $\mu(\lambda)$ is decreasing.

It remains to prove (2.14). Since $\lambda_1[A; B] \geq 0$, it is clear that the unique root of

$m(\lambda) = \lambda_1[1; \frac{B}{A}] - \frac{1}{A}$

is non-negative, in fact, it is $\lambda_1[A; B]$. But, $m(\lambda)$ is decreasing, and so $m(0) \geq 0$, that is, $\lambda_1[1; \frac{B}{A}] \geq 0$.

(III) Its proof is analogous to paragraph (II).

To avoid confusion we denote

$$\lambda_1(B) := \lambda_1[1; B], \quad \lambda_1 := \lambda_1(0).$$

**Remark 2.2.** In Lemma 2.3 of [18] it is assured that $\lambda_1[A; B]$ is increasing in $A$ for all $B$, but the proof does not seem to be correct.
For each \( m \in L^\infty(\Omega) \) we denote by \( \xi[m] \) the unique solution of

\[
\begin{cases}
-\Delta \xi = m(x) & \text{in } \Omega, \\
\xi = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(2.15)

It is clear that if \( m \geq 0, m \not\equiv 0 \) then \( \xi[m] > 0 \), that the map \( m \mapsto \xi[m] \) is increasing and that for every positive constant \( R > 0 \) there holds \( \xi[m] = R\xi[m/R] \).

When \( v \equiv 0 \), then \( u \) verifies (1.5). When \( u \) is zero, \( v \) verifies a logistic equation of the following type

\[
\begin{cases}
-\Delta[(1 + a(x) + \alpha u) u] = u(\lambda + m(x) - u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(2.16)

where \( a \in C^2(\overline{\Omega}), m \in C(\overline{\Omega}) \) and \( a, m \geq 0, \alpha > 0 \) and \( \lambda \in \mathbb{R} \). Also, along the paper we use the solution of the following equation

\[
\begin{cases}
-\Delta[(1 + a(x) + \alpha u) u] = u(\lambda + m(x)) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(2.17)

First, we study (2.17).

**Proposition 2.3.** If \( \lambda > \lambda_1[1 + a; -m] \) then there exists a positive solution of (2.17). Denoting by \( u_\alpha \) any positive solution of (2.17), the following equality holds

\[
\alpha u_\alpha = u_1.
\]  

(2.18)

Assume now

\[
\lambda + m(x) > 0 \quad \text{for all } x \in \Omega,
\]  

(2.19)

then there exists a positive solution of (2.17) if and only if \( \lambda > \lambda_1[1 + a; -m] \). Moreover, the positive solution \( u_\alpha \) is unique and stable.

**Proof.** To prove the existence, we use the sub-supersolution method. Denote by \( \varphi_1 > 0 \) the principal eigenfunction associated to \( \lambda_1[1 + a; -m] \) such that \( \|\varphi_1\|_{L^\infty(\Omega)} = 1 \). Then, \( u = \varepsilon \varphi_1 \) is a subsolution if

\[
2\alpha \varepsilon (-|\nabla \varphi_1|^2 + \varphi_1(-\Delta \varphi_1)) \leq \varphi_1(\lambda - \lambda_1[1 + a; -m]).
\]  

(2.20)
It is enough to take \( \varepsilon > 0 \) small because \( \lambda > \lambda_1[1 + a; -m] \).

We construct now a supersolution. Take \( \Omega \subset \tilde{\Omega} \) and \( \tilde{\xi}[1] \) the solution of (2.15) in \( \tilde{\Omega} \).

Take as supersolution \( u = u_K \) with \( K > 0 \) such that
\[
K\tilde{\xi}[1] = u_K(1 + a(x) + \alpha u_K) \iff u_K = \frac{2K\tilde{\xi}[1]}{\sqrt{(1 + a)^2 + 4\alpha K\tilde{\xi}[1] + (1 + a)}}.
\]
Then, \( u_K \) is a supersolution provided that
\[
-\Delta[(1 + a(x) + \alpha u_K) u_K] = K \geq u_K(\lambda + m(x)) \iff
K \geq (\lambda + m(x)) \frac{2K\tilde{\xi}[1]}{\sqrt{(1 + a)^2 + 4\alpha K\tilde{\xi}[1] + (1 + a)}}.
\]

It is enough take \( K > 0 \) large. Also, for \( \varepsilon \) small and \( K \) large, the sub and supersolutions are ordered. Then, there exists a solution \( u \) such that \( \underline{u} \leq u \leq \bar{u} \).

Assume now (2.19). If \( u \) is a positive solution of (2.17) corresponding to \( \lambda \), then
\[
\lambda_1[1 + a(x) + \alpha u; -\lambda - m] = 0.
\]
Then, using (2.19) and Proposition 2.1 (III) we get
\[
0 > \lambda_1[1 + a; -\lambda - m],
\]
or equivalently, \( \lambda > \lambda_1[1 + a; -m] \).

Suppose that (2.17) possesses two solutions \( u_1 \neq u_2 \). Denote by \( w = u_1 - u_2 \), then
\[
-\Delta[(1 + a(x) + \alpha u_1) u_1 - (1 + a(x) + \alpha u_2) u_2] = (\lambda + m)(u_1 - u_2).
\]
Thus
\[
-\Delta[(1 + a(x) + \alpha(u_1 + u_2)) w] = (\lambda + m) w,
\]
and then for some \( j \geq 1 \)
\[
0 = \lambda_j[1 + a + \alpha(u_1 + u_2); -\lambda - m] \geq \lambda_1[1 + a + \alpha(u_1 + u_2); -\lambda - m],
\]
and so using Proposition 2.1 (III) we get \( 0 > \lambda_1[1 + a(x) + \alpha u_1; -\lambda - m] \), a contradiction with (2.21).

To prove the stability for a solution \( u > 0 \), we need to show that
\[
\lambda_1[1 + a + 2\alpha u; -\lambda - m] > 0.
\]
It suffices to apply (2.21) again and Proposition 2.1 (II).

Finally, observe that \( \alpha u_\alpha \) verifies (2.17) with \( \alpha = 1 \).
For a continuous function $f \in C(\Omega)$, we denote

$$f_M := \sup_{x \in \Omega} f(x), \quad f_L := \inf_{x \in \Omega} f(x).$$

With respect to (2.16) we have

**Proposition 2.4.** If $\lambda > \lambda_1[1 + a; -m]$ then (2.16) possesses at least a positive solution. Moreover, the following estimate holds for every solution $u$ of (2.16)

$$\|u\|_{L^\infty(\Omega)} \leq \lambda + m_M. \quad (2.23)$$

Assume now that

$$\alpha(\lambda + m(x)) + 1 + a(x) > 0 \quad \text{for all } x \in \Omega, \quad (2.24)$$

then there exists a positive solution of (2.16) if and only if $\lambda > \lambda_1[1 + a; -m]$. Moreover, the positive solution is unique and stable.

**Proof.** Since the proof is rather similar to the last Proposition, we will be brief in some steps.

To prove the existence, we use again the sub-supersolution method. It is clear that $u = \varepsilon \varphi_1$ and $\overline{u} = u_\alpha$ are respectively sub and supersolution of (2.16) where $u_\alpha$ is a positive solution of (2.17) and $\varepsilon > 0$ is small enough.

We show now (2.23). Define

$$\Omega_1 := \{x \in \Omega : u(x) > \lambda + m_M\},$$

and assume that $\Omega_1 \neq \emptyset$. Then

$$-\Delta[(1 + a(x) + \alpha u)u] \leq 0 \quad \text{in } \Omega_1.$$ 

Then, the maximum of $(1 + a(x) + \alpha u)u$ is attained in $\partial \Omega_1$, that is

$$(1 + a(x) + \alpha u(x))u(x) \leq (1 + a(x) + \alpha(\lambda + m_M))(\lambda + m_M),$$

that is $u \leq \lambda + m_M$ in $\Omega_1$, an absurdum. Then $\Omega_1 = \emptyset$.

Assume now (2.24). It is clear that if $u$ is a solution corresponding to $\lambda$, then

$$\lambda_1[1 + a + \alpha u; -\lambda - m + u] = 0. \quad (2.25)$$
and so, using (2.24) and Proposition 2.1 (III) we get that \( 0 > \lambda_1[1 + a(x); -\lambda - m] \).

Suppose that (2.16) possesses two solutions \( u_1 \neq u_2 \). Denote by \( w = u_1 - u_2 \neq 0 \), then

\[
-\Delta[(1 + a(x) + \alpha u_1)u_1 - (1 + a(x) + \alpha u_2)u_2] = u_1(\lambda + m - u_1) - u_2(\lambda + m - u_2).
\]

Hence

\[
-\Delta[(1 + a(x) + \alpha (u_1 + u_2))w] = (\lambda + m - (u_1 + u_2))w,
\]

and then for some \( j \geq 1 \) we get that \( \lambda_j[1 + a(x) + \alpha(u_1 + u_2); -\lambda - m + u_1 + u_2] = 0 \). Then,

\[
\lambda_1[1 + a(x) + \alpha(u_1 + u_2); -\lambda - m + u_1 + u_2] \leq 0. \tag{2.26}
\]

Now, using (2.24) we have that

\[
-\frac{-\lambda - m + u_1 + u_2}{1 + a + \alpha(u_1 + u_2)} > -\frac{-\lambda - m + u_1}{1 + a + \alpha u_1}.
\]

Then, by (2.25) we can apply Proposition 2.1 (III) and conclude that

\[
\lambda_1[1 + a(x) + \alpha(u_1 + u_2); -\lambda - m + u_1 + u_2] > 0,
\]

a contradiction with (2.26).

To prove the stability, we need to show that

\[
\lambda_1[1 + a(x) + 2\alpha u; -\lambda - m + 2u] > 0.
\]

It suffices to apply again Proposition 2.1 (II). This completes the proof.

\[\square\]

**Remark 2.5.** (I) Assume that \( m \) is constant. In this case, \( \lambda > \lambda_1[1 + a; -m] \) is a necessary and sufficient condition for the existence of positive solution of (2.17). Indeed, if \( \lambda + m \leq 0 \) the by the maximum principle we have that the unique solution is the trivial one. If \( \lambda + m > 0 \), the result follows by Proposition 2.3.

Analogous result for the equation (2.16).

(II) Equation (2.16) was studied in [16] only when \( m \equiv 0 \). Observe that in this case \( \lambda > 0 \) is a necessary condition to (2.16) possesses a positive solution. Therefore condition (2.24) is satisfied. In this sense, our Proposition 2.4 generalizes Theorem 2.1 in [16]. See also Theorem 2.11 in [17] for \( m \equiv a \equiv 0 \) and Robin boundary conditions. On the other hand, in Theorem 2.10 in [18] the above result was proved, again under Robin boundary conditions, showing the uniqueness of positive solution without condition (2.24). However, in their proof they used that \( \lambda_1[A; B] \) is increasing in \( A \) for any \( B \), see Remark 2.2. We conjecture that the uniqueness of positive solution follows without condition (2.24).
3 The first a priori bounds of coexistence states.

As we pointed out in the Introduction of the paper, the semi-trivial solutions $(\theta_\lambda, 0)$ and $(0, \theta_{\mu, \alpha})$ determine $(\lambda, \mu)$-regions where only one of the species lives. Remember that $\theta_\lambda$ and $\theta_{\mu, \alpha}$ are stable and so

$$\lambda_1 (2\theta_\lambda - \lambda) > 0 \quad \text{and} \quad \lambda_1 [1 + 2\alpha \theta_{\mu, \alpha}; 2\theta_{\mu, \alpha} - \mu] > 0.$$  \hfill (3.27)

As consequence of (2.18), we have the following result related to the semi-trivial solutions.

**Proposition 3.1.** Assume that $\mu > \lambda_1$. Then:

(I) $\|\theta_{\mu, \alpha}\|_{L^\infty(\Omega)} \to 0$ as $\alpha \to +\infty$.

(II) $\theta_{\mu, \alpha} \to \theta_\mu$ in $C^2(\Omega)$ as $\alpha \to 0$.

**Proof.** (I) It is clear that $\theta_{\mu, \alpha} \leq \omega_{\mu, \alpha}$ where $\omega_{\mu, \alpha}$ is the solution of

$$-\Delta((1 + \alpha w)w) = \mu w \quad \text{in} \quad \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega.$$  

By (2.18) we get that $\omega_{\mu, \alpha} = \frac{\omega_{\mu, 1}}{\alpha}$. This completes the proof of the first paragraph.

(II) By (2.23) we know that $\theta_{\mu, \alpha} \leq \mu$. On the other hand, by a similar argument to the used in (2.20), $\varepsilon \varphi_1$ is subsolution of (1.6) if

$$\lambda_1 \varphi_1 + 2\alpha \varepsilon (-|\nabla \varphi_1|^2 + \lambda_1 \varphi_1^2) \leq \varphi_1 (\mu - \varepsilon \varphi_1),$$

for which it is sufficient that

$$\varepsilon (2\alpha \lambda_1 + 1) \leq \mu - \lambda_1.$$  

Then, $$(\mu - \lambda_1)/(2\alpha \lambda_1 + 1) \varphi_1 \leq \theta_{\mu, \alpha} \quad \text{in} \quad \Omega,$$ and so $\theta_{\mu, \alpha}$ is far from zero as $\alpha \to 0$. Hence, using elliptic arguments, we obtain the result. \qed

**Remark 3.2.** Observe that when the self-diffusion is large, the species goes to the extinction.

To find regions of coexistence states and non-existence of coexistence states of (1.1), it is useful to perform the change of variable

$$w = (1 + \alpha v + \beta u)v \iff v = \varphi(u, w) := \frac{2w}{\sqrt{(1 + \beta u)^2 + 4\alpha w + (1 + \beta u)}},$$  \hfill (3.28)
A symbiotic self-cross diffusion model

which transforms system (1.1) into

\[
\begin{align*}
-\Delta u &= u(\lambda - u + b\varphi(u, w)) \quad \text{in } \Omega, \\
-\Delta w &= \varphi(u, w)(\mu - \varphi(u, w) + cu) \quad \text{in } \Omega, \\
u = w &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(3.29)

In some cases, it is convenient to write

\[\varphi(u, w) := wf(u, w), \quad \text{where } f(u, w) := \frac{2}{\sqrt{(1 + \beta u)^2 + 4\alpha w + (1 + \beta u)}}.\]

(3.30)

Of course, the study of (3.29) is equivalent to (1.1). Observe that (3.29) possesses the trivial solution \((0, 0)\) and the semitrivial solutions \((u, w) = (\theta \lambda, 0)\) and \((u, w) = (0, \Theta_{\mu, \alpha})\) where \(\Theta_{\mu, \alpha}\) is the unique solution of

\[
\begin{align*}
-\Delta w &= \varphi(0, w)(\mu - \varphi(0, w)) \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(3.31)

It is clear that

\[\theta_{\mu, \alpha} = 2 \frac{\Theta_{\mu, \alpha}}{\sqrt{1 + 4\alpha \Theta_{\mu, \alpha} + 1}}.\]

Proposition 3.3. Let \((u, v)\) be a coexistence state of (1.1). Then,

(I) \(\theta_{\lambda} \leq u \leq u_M \leq \lambda + bu_M\).

(II) \(v_M \leq \mu + cu_M\).

(III) \(w(x) \leq \xi_{\mu + cu/4}(x)\) for all \(x \in \Omega\).

Proof. Let \(x_M, z_M \in \Omega\) be points such that \(u_M := u(x_M) = \sup_{\Omega} u(x)\) and \(v_M := v(z_M) = \sup_{\Omega} v(x)\). By the maximum principle we can assert that \(-\Delta u(x_M) \geq 0\) and \(-\Delta v(z_M) \geq 0\). (I) follows directly. (II) follows by (2.23). Finally, since

\[\varphi(u, w)(\mu - \varphi(u, w) + cu) \leq \frac{(\mu + cu)^2}{4}\]

we obtain (III) from (3.29). \(\square\)
We can establish the following region of non-existence and a priori bounds for coexistence states in the case $bc < 1$ for every $\alpha > 0$ and $\beta > 0$. The proof follows from Proposition 3.3.

**Proposition 3.4.** Assume that $bc < 1$.

(I) If $(u, v)$ is a coexistence state of (1.1), we have

$$u \leq \frac{\lambda + b\mu}{1 - bc} \quad \text{and} \quad v \leq \frac{\mu + c\lambda}{1 - bc}.$$

(II) If there exists a coexistence state of (1.1), then

$$\lambda + b\mu > 0 \quad \text{and} \quad \mu + c\lambda > 0.$$

4 A priori-bounds.

In this Section we prove the existence of a priori bounds for the coexistence states of (1.1) for the case $b > 0$, $c > 0$ and $\beta$ or $\alpha$ large.

**Proposition 4.1.** Assume that for any fixed positive number $R > 0$

$$\max \{|\lambda|, |\mu|\} \leq R.$$

(I) There exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$ a constant $C = C(R, \alpha, \beta, \Omega, b, c)$ exists such that

$$\|u\|_{L^\infty(\Omega)} \leq C, \quad \|w\|_{L^\infty(\Omega)} \leq C,$$

for every coexistence state $(u, w)$ of (3.29).

(II) There exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$ a constant $C = C(R, \alpha, \beta, \Omega, b, c)$ exists such that

$$\|u\|_{L^\infty(\Omega)} \leq C, \quad \|w\|_{L^\infty(\Omega)} \leq C,$$

for every coexistence state $(u, w)$ of (3.29).

**Proof.** We will use the blow-up argument of Gidas-Spruck, [4]. For completeness, we give the proof of the result.

(I) To prove this item we assume that there exist a sequence $(\beta_n, \lambda_n, \mu_n)$ with $|\lambda_n| \leq R$, $|\mu_n| \leq R$, $\beta_n \to \infty$ and a sequence of coexistence states $(u_n, w_n)$ of (3.29) such that

$$\|u_n\|_{L^\infty(\Omega)} + \|w_n\|_{L^\infty(\Omega)} \to \infty.$$ 

It is clear that if $\|u_n\|_{L^\infty(\Omega)} \to \infty$ then $\|w_n\|_{L^\infty(\Omega)} \to \infty$.
by Proposition 3.3. Now, suppose that \( \|w_n\|_{L^\infty(\Omega)} \to \infty \) and \( \|u_n\|_{L^\infty(\Omega)} \leq C \). Then, by Proposition 3.3 (III)

\[
w_n \leq \xi[(\mu_n + c u_n)^2/4] \leq \xi[(\mu_n + c C)^2/4] \leq C
\]

then \( w_n \) is bounded, a contradiction.

Denote by

\[
M_n := \|u_n\|_{L^\infty(\Omega)} = u(x_n) = \max_{x \in \Omega} u_n(x)
\]

for some \( x_n \in \Omega \), thus \( M_n \to \infty \). By the compactness of \( \overline{\Omega} \) we can assume that \( x_n \to x_0 \in \overline{\Omega} \). We distinguish two cases.

Case 1: \( x_0 \in \Omega \). Define

\[
\delta := \frac{\text{dist}(x_0, \partial \Omega)}{2} > 0.
\]

We make now the following change of variable

\[
U_n(y) := \frac{u_n(y M_n^{-1/2} + x_n)}{M_n}, \quad W_n(y) := \frac{w_n(y M_n^{-1/2} + x_n)}{M_n^2}
\]

in \( \Omega_n \),

where \( \Omega_n := \{y \in \mathbb{R}^N : y M_n^{-1/2} + x_n \in \Omega\} \).

Observe that if \( |y| < \delta M_n^{1/2} \) then \( y M_n^{-1/2} + x_n \in \Omega \). Thus given \( S > 0 \) there exists \( n \in \mathbb{N} \) large enough such that \( B(0, S) \subset B(0, \delta M_n^{1/2}) \), where \( B(0, S) \) stands for the ball of radius \( S > 0 \) centered at the origin.

Observe that

\[
\|U_n\|_{L^\infty(B(0,S))} = 1 \quad \text{and} \quad U_n(0) = 1.
\]

On the other hand, from Proposition 3.3 we get that

\[
w_n \leq \xi[(\mu_n + c u_n)^2/4] \leq \xi[(\mu_n + c C)^2/4],
\]

and hence,

\[
W_n = \frac{w_n}{M_n^2} \leq \xi[(\mu_n + c M_n)^2/4] \leq C
\]

for some \( C > 0 \) and \( n \) large. Hence,

\[
\|W_n\|_{L^\infty(B(0,S))} \leq C.
\]

The pair \( (U_n, W_n) \) satisfies the following system

\[
\begin{cases}
-\Delta U_n = \mathcal{F}(U_n, W_n) \quad \text{in } B(0, S), \\
-\Delta W_n = \mathcal{G}(U_n, W_n) \quad \text{in } B(0, S),
\end{cases}
\]

(4.32)
where

\[ F(U_n, W_n) := \lambda_n M_n^{-1} U_n - U_n^2 + \frac{2b}{\beta_n} \frac{U_n W_n}{\sqrt{\left(\frac{1}{\beta_n M_n} + U_n\right)^2 + 4 \frac{\alpha_n}{\beta_n^2} W_n + \frac{1}{M_n \beta_n} + U_n}}, \]

and

\[ G(U_n, W_n) := \frac{2W_n}{\sqrt{(1 + \beta_n M_n U_n)^2 + 4 \alpha M_n^2 W_n + 1 + \beta_n M_n U_n}} \cdot \left( \mu_n M_n^{-1} - \frac{2M_n W_n}{\sqrt{(1 + \beta_n M_n U_n)^2 + 4 \alpha M_n^2 W_n + 1 + \beta_n M_n U_n}} + cU_n \right). \]

Since \( U_n \) and \( W_n \) are bounded in \( L^\infty(B(0,S)) \), then

\[ \|F(U_n, W_n)\|_{L^\infty(B(0,S))} \leq C, \]

hence \( U_n \) is bounded in \( C^{1,\nu}(\overline{B(0,S)}) \) for some \( 0 < \nu < 1 \), which provides bounds in \( C^{2,\nu}(\overline{B(0,S)}) \). Observe also that

\[ \frac{2b}{\beta_n} \frac{U_n W_n}{\sqrt{\left(\frac{1}{\beta_n M_n} + U_n\right)^2 + 4 \frac{\alpha_n}{\beta_n^2} W_n + \frac{1}{M_n \beta_n} + U_n}} \leq \frac{2b}{\beta_n} W_n \to 0 \quad \text{as} \quad n \to \infty. \]

We can pass to the limit in the first equation, and after a standard argument we have that \( U_n \to U \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \), then \( U \) is a solution of

\[ -\Delta U = -U^2 \quad \text{in} \quad \mathbb{R}^N, \quad (4.33) \]

with \( 0 \leq U \leq 1 \), \( U(0) = 1 \). This implies that \( U \equiv 0 \), a contradiction.

Case 2: \( x_0 \in \partial \Omega \). Observe that in this case, \( \Omega_n \to \mathbb{R}_+^N \). After a linear change of variable, we arrive at the equation

\[ \begin{cases} 
-\Delta U = -U^2 \quad \text{in} \quad \mathbb{R}_+^N, \\
U = 0 \quad \text{in} \quad \partial \mathbb{R}_+^N,
\end{cases} \quad (4.34) \]

for some regular and bounded non-negative function with \( U(0) = 1 \), again a contradiction.

(II) The second item follows by a similar argument. Indeed, in this case \( \beta \) is fixed and take \( \alpha_n \to \infty \). In this case,

\[ F(U_n, W_n) = \lambda_n M_n^{-1} U_n - U_n^2 + \frac{bU_n}{\alpha_n^{1/2}} \frac{2W_n}{\sqrt{\left(\frac{1}{\alpha_n} \left(\frac{1}{M_n} + \beta U_n\right)^2 + 4 \alpha_n W_n + \frac{1}{M_n \alpha_n^{1/2}} + \frac{\beta^2}{\alpha_n^{1/2}} U_n}}}, \]
and as \( n \to \infty \)

\[
\frac{bU_n}{\alpha_n^{1/2}} \sqrt{\frac{1}{\alpha_n} \left( \frac{1}{M_n} + \beta U_n \right)^2 + 4W_n + \frac{1}{M_n \alpha_n^{1/2}} + \frac{\beta}{\alpha_n^{1/2}} U_n} \leq \frac{bU_n}{\alpha_n^{1/2}} (1 + W_n^{1/2}) \to 0.
\]

We can continue exactly the same lines as paragraph (I).

\[\square\]

5 Existence of solutions.

In this Section we prove Theorem 1.1. First we need to establish some properties of \( F \) and \( G \), defined in (1.7) and (1.8).

**Lemma 5.1.** (I) Fix \( \alpha > 0 \). Then, \( F \) is a decreasing map in \( \mu \) and \( \lim_{\mu \to +\infty} F(\alpha, \mu) = -\infty \). Fix now \( \mu > \mu_1 \), then \( \lim_{\alpha \to 0} F(\alpha, \mu) = \lambda_1(-b\theta_\mu) \) and \( \lim_{\alpha \to +\infty} F(\alpha, \mu) = \lambda_1 \).

(II) Fix \( \beta \geq 0 \). Then,

(a) If \( \beta \lambda_1 > c \), then \( G \) is increasing in \( \lambda \) and \( \lim_{\lambda \to +\infty} G(\beta, \lambda) = +\infty \).

(b) If \( \beta \lambda_1 = c \), then \( G(\beta, \lambda) = \lambda_1 \).

(c) If \( \beta \lambda_1 < c \), then \( G \) is decreasing in \( \lambda \) and \( \lim_{\lambda \to +\infty} G(\beta, \lambda) = -\infty \).

(III) Fix \( \lambda > \lambda_1 \). Then \( G \) is increasing in \( \beta \), and

\[
\lim_{\beta \to 0} G(\beta, \lambda) = \lambda_1(-c\theta_\lambda) \quad \text{and} \quad \lim_{\beta \to +\infty} G(\beta, \lambda) = +\infty.
\]

**Proof.** Assertions (II) and (III) have been proved in [2]. Item (I) follows because \( \mu \mapsto \theta_{\mu, \alpha} \) is increasing and by Proposition 3.1. \[\square\]

We state now a result showing the stability of the trivial and semi-trivial solutions of (1.1). Its proof is rather similar to Proposition 4.1 in [3], so we omit it.

**Proposition 5.2.** (I) The trivial solution of (1.1) is linearly asymptotically stable if \( \lambda < \lambda_1 \) and \( \mu < \lambda_1 \) and unstable if \( \lambda > \lambda_1 \) or \( \mu > \lambda_1 \).

(II) Assume that \( \lambda > \lambda_1 \). The semi-trivial solution \( (\theta_\lambda, 0) \) is linearly asymptotically stable if \( \mu < G(\beta, \lambda) \) and unstable if \( \mu > G(\beta, \lambda) \).

(III) Assume that \( \mu > \lambda_1 \). The semi-trivial solution \( (0, \theta_{\mu, \alpha}) \) is linearly asymptotically stable if \( \lambda < F(\mu, \alpha) \) and unstable if \( \lambda > F(\mu, \alpha) \).
Proof of Theorem 1.1. We begin with the nonexistence results (I) and (II). Let $\varphi_1$ be a positive eigenfunction associated to $\lambda_1$. If we multiply the first equation of (1.1) by $k\varphi_1$ and the second one by $\varphi_1$, integrate and add both equations, we obtain

$$
\int_\Omega \varphi_1 [(\lambda_1 - \lambda)ku + (\lambda_1 - \mu)v] = \int_\Omega \varphi_1^2 \left[ -(1 + \alpha \lambda_1) \left( \frac{v}{u} \right)^2 + (kb + c - \beta \lambda_1) \frac{v}{u} - k \right].
$$

(5.35)

Denote $f(r) = -(1 + \alpha \lambda_1)r^2 + (kb + c - \beta \lambda_1)r - k$.

Hence in situation (I), if $\beta \lambda_1 \geq c$, then for $k = 0$, $f(r) < 0$ for every $r > 0$. The first member of (5.35) is negative, then $\lambda_1 - \mu < 0$.

In item (II) we look for $k > 0$ such that $f(r) < 0$ for every $r > 0$. Since $f(0) < 0$, it is enough to find $k > 0$ such that $(kb + c - \beta \lambda_1)^2 - 4k(1 + \alpha \lambda_1) < 0$. It is easy to see that this is reached if $b(c - \beta \lambda_1) < 1 + \alpha \lambda_1$. Then, if $\lambda, \mu \leq \lambda_1$ there does not exist any coexistence state of (1.1).

We work with the transformed system (3.29) which is equivalent to (1.1). In order to get the existence results (III) to (V) we use the fixed point index with respect to the positive cone, see [1] for the following notations and result. Define

$$
X = \{u \in C(\Omega) : u = 0 \text{ on } \partial \Omega\}, \quad P := \{u \in X : u \geq 0 \text{ in } \Omega\},
$$

$$
E = X \oplus X, \quad W = P \oplus P.
$$

For $y \in W$, define

$$
W_y = \{x \in E : y + tx \in W \text{ for some } t > 0\}, \quad S_y = \{x \in W_y : -x \in W_y\}.
$$

Assume $T : W \mapsto W$ is a compact operator and $T(y_0) = y_0$ for some $y_0 \in W$. We define

$$
L := DT(y_0),
$$

where $DT(y_0)$ denotes the Fréchet derivative of $T$ at $y_0$. We say that $L$ has property $\alpha$ if there exist $t \in (0, 1)$ and $w \in W_y \setminus S_y$ such that $w - tLw \in S_y$. The following result follows by [1] and [9].

**Theorem 5.3.**

1. If $I - L$ is invertible in $E$, $L$ has the property $\alpha$ on $W_{y_0}$, then $i_W(T, y_0) = 0$. 

2. If $I - L$ is invertible in $E$, $L$ does not have the property $\alpha$ on $\overline{W_{y_0}}$, then $i_W(T, y_0) = (-1)^\sigma$, where $\sigma$ is the multiplicity of eigenvalues of $L$ greater than 1.

3. If $I - L$ is not invertible in $E$ but invertible on $\overline{W_{y_0}}$, and $I - L$ is not surjective from $\overline{W_{y_0}}$ to $\overline{W_{y_0}}$, then $i_W(T, y_0) = 0$.

Assume that $bc < 1$ or $\beta, \alpha$ such that $b(c - \beta \lambda_1) < 1 + \alpha \lambda_1$, then by (II), (3.29) does not possess positive solution for $\lambda, \mu \leq \lambda_1$. Fix such values of $\alpha$ and $\beta$.

Take $a < \lambda_1$, $t \in [0, 1]$ and consider the following system

$$
\begin{cases}
-\Delta u = u(t\lambda + (1 - t)a - u + b\varphi(u, w)) & \text{in } \Omega, \\
-\Delta w = \varphi(u, w)(t\mu + (1 - t)a - \varphi(u, w) + cu) & \text{in } \Omega, \\
u = w = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Observe that if $bc < 1$ or $\beta \geq \beta_0$ or $\alpha \geq \alpha_0$, where $\beta_0$ and $\alpha_0$ are the constants of Proposition 4.1, also recall Proposition 3.4, then, there exist $R_1$ and $R_2$ such that

$$
\|u\|_{L^\infty(\Omega)} \leq R_1, \quad \|w\|_{L^\infty(\Omega)} \leq R_2.
$$

Note that $R_1$ and $R_2$ depend on $|\lambda|, |\mu|, a, b, c, \alpha, \beta$ and $\Omega$ but they are independent of $t$.

Finally, take $M > 0$ large enough such that

$$
M > \max\{|\lambda| + R_1, \|f(u, w)\|_\infty(|\mu| + \|\varphi(u, w)\|_\infty),
$$

where $f$ and $\varphi$ are defined in (3.30) and

$$
\|f(u, w)\|_\infty = \max\{|f(u, w)|; 0 \leq u \leq R_1, 0 \leq w \leq R_2\},
$$

$$
\|\varphi(u, w)\|_\infty = \max\{|\varphi(u, w)|; 0 \leq u \leq R_1, 0 \leq w \leq R_2\}.
$$

Define the set

$$
S := \{(u, v) \in E : 0 \leq u \leq R_1 + 1, 0 \leq w \leq R_2 + 1\},
$$

and the operators $\mathcal{H} : [0, 1] \times E \mapsto E$ defined by

$$
\mathcal{H}(t, u, w) := \begin{pmatrix}
T(u(t\lambda + (1 - t)a + M - u + b\varphi(u, w)) \\
T((\mu t + (1 - t)a)\varphi(u, w) + Mw - \varphi^2(u, w) + cu\varphi(u, w))
\end{pmatrix},
$$
where \( T := (-\Delta + M)^{-1} \) stands for the inverse of the operator \(-\Delta + M\) in \( \Omega \) under homogeneous Dirichlet boundary conditions. Define

\[
\mathcal{K} := \mathcal{H}(1, u, w).
\]

It is clear that \( \mathcal{H}(t, \cdot) \) is a continuous and compact operator and \( \mathcal{H}(t, \cdot)(S) \subset W \).

We claim that if \((\lambda, \mu)\) verifies (1.9) then

\[(I.1)\ i_W(\mathcal{K}, \text{int}(S)) = 1;\]

\[(I.2)\ i_W(\mathcal{K}, (0, 0)) = 0 \text{ if } \lambda > \lambda_1 \text{ or } \mu > \lambda_1;\]

\[(I.3)\ i_W(\mathcal{K}, (\theta\lambda, 0)) = i_W(\mathcal{K}, (0, \Theta\mu, \alpha)) = 0.\]

\[(I.1)\] It follows by homotopy invariance that

\[
i_W(\mathcal{K}, \text{int}(S)) = i_W(\mathcal{H}(1, \cdot), \text{int}(S)) = i_W(\mathcal{H}(0, \cdot), \text{int}(S)) = i_W(\mathcal{H}(0, \cdot), (0, 0)),
\]

this last inequality follows because \( a < \lambda_1 \), and then the unique positive solution of \( \mathcal{H}(0, \cdot) \) is the trivial solution. We show now that

\[
i_W(\mathcal{H}(0, \cdot), (0, 0)) = 1.
\]

Observe that

\[
D\mathcal{H}(0, \cdot)(0, 0) \left( \begin{array}{c} \xi \\ \eta \end{array} \right) = \left( \begin{array}{c} T(M + a)\xi \\ T(M + a)\eta \end{array} \right).
\]

Since \( a < \lambda_1 \), then \( I - D\mathcal{H}(0, \cdot)(0, 0) \) is invertible in \( E \).

On the other hand, \( W_{(0,0)} = W \), \( S_{(0,0)} = \{(0,0)\} \) and it is not hard to show that \( D\mathcal{H}(0, \cdot)(0, 0) \) does not have property \( \alpha \) and there is no eigenvalue of \( D\mathcal{H}(0, \cdot) \) greater than 1, see page 1092 in [13]. Then, the result concludes applying 2 of Theorem 5.3.

\[(I.2)\] Assume that \( \lambda > \lambda_1 \) or \( \mu > \lambda_1 \). Observe that

\[
D\mathcal{K}(0, 0) \left( \begin{array}{c} \xi \\ \eta \end{array} \right) = \left( \begin{array}{c} T(M + \lambda)\xi \\ T(M + \mu)\eta \end{array} \right).
\]

Then, the above relation implies that \( I - D\mathcal{K}(0, 0) \) is invertible in \( W_{(0,0)} = W \). Moreover, it is not surjective from \( W \) to \( W \). For that, consider for instance \( \lambda > \lambda_1 \) and take \( h > 0 \),
then \((T(h), 0)^t \not\in \text{Rg}(I - DK(0, 0))\). Indeed, assume that there exists \((\varphi_1, \varphi_2)^t \in W\) such that
\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} - DK(0, 0)
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} = \begin{pmatrix}
T(h) \\
0
\end{pmatrix},
\]
and so,
\[
(-\Delta - \lambda)\varphi_1 = h \quad \text{in } \Omega, \quad \varphi_1 = 0 \quad \text{on } \partial\Omega.
\]
This is an absurdum. Then, applying 3 of Theorem 5.3 we conclude that \(i_W(K, (0, 0)) = 0\).

(I.3) Assume that \((\lambda, \mu)\) verifies (1.9). In this case,
\[
L = DK(\theta \lambda, 0)
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix} = \begin{pmatrix}
T((M + \lambda - 2\theta \lambda)\xi + b\theta \lambda \varphi_w(\theta \lambda, 0)\eta) \\
T((M + \mu + \theta \lambda \frac{c\theta \lambda}{1 + \beta \theta \lambda})\eta)
\end{pmatrix}.
\]
In this case, \(W_{(\theta \lambda, 0)} = X \oplus P\) and \(S_{(\theta \lambda, 0)} = X \oplus \{0\}\). We claim that \(I - L\) is invertible in \(W_{(\theta \lambda, 0)}\). Otherwise there exists \((\xi, \eta)^t \in W_{(\theta \lambda, 0)}\) such that
\[
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix} = L
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}.
\] (5.38)
From the second equation,
\[
-\Delta \eta - \frac{\mu + c\theta \lambda}{1 + \beta \theta \lambda} \eta = 0 \quad \text{in } \Omega, \quad \eta = 0 \quad \text{on } \partial\Omega.
\]
Since \(\eta > 0\) we can conclude that
\[
\lambda_1 \left( -\frac{\mu + c\theta \lambda}{1 + \beta \theta \lambda} \right) = 0.
\] (5.39)
Observe that if \((\lambda, \mu)\) verifies (1.9), then \(\mu > \lambda_1[1 + \beta \theta \lambda; -c\theta \lambda]\), or equivalently \(0 > \lambda_1[1 + \beta \theta \lambda; -\mu - c\theta \lambda]\), which implies
\[
\lambda_1 \left( -\frac{\mu + c\theta \lambda}{1 + \beta \theta \lambda} \right) < 0,
\] (5.40)
which is an absurdum with (5.39).
i) Assume now that \(I - L\) is invertible in \(E\). We show now that \(L\) has the property \(\alpha\). Indeed, \(L\) has the property \(\alpha\) if there exist \(t \in (0, 1)\) and \((\xi, \eta)^t \in X \oplus P \setminus X \oplus \{0\}\) such
that
\[
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
- tL
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
\in X \oplus \{0\}.
\]
It suffices to find \( t \in (0, 1) \) and \( \eta \in P \setminus \{0\} \) such that
\[
\begin{cases}
(\Delta + M)\eta = t(M + \frac{\mu + c\theta_\lambda}{1 + \beta\theta_\lambda})\eta & \text{in } \Omega, \\
\eta = 0 & \text{on } \partial\Omega,
\end{cases}
\]
or equivalently, to find \( t \in (0, 1) \) such that
\[
r(t) = \lambda_1(M(1-t) - i\frac{\mu + c\theta_\lambda}{1 + \beta\theta_\lambda}) = 0.
\]
Observe that \( r(0) > 0 \) and \( r(1) < 0 \) by (5.40), then the result follows. Hence, in this case by 1 of Theorem 5.3, \( i_W(K, (\theta_\lambda, 0)) = 0 \).

ii) Assume now that \( I - L \) is not invertible in \( E \), then there exists \((\xi, \eta)^t \in E \setminus \{(0, 0)\}\) such that (5.38). We distinguish two cases:

1.- Assume \( \eta \equiv 0 \). Then, by the first equation
\[
-\Delta \xi + (2\theta_\lambda - \lambda)\xi = 0.
\]
Since \( \theta_\lambda \) is stable, so (see 3.27) \( \lambda_1(2\theta_\lambda - \lambda) > 0 \). Hence, \( \xi \equiv 0 \).

2.- Assume that \( \eta \neq 0 \). Consider \( \varphi \in P \) such that
\[
\int_\Omega \eta \varphi \neq 0,
\]
and \( \psi > 0 \) such that
\[
(\Delta + M)\psi = \varphi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega.
\]
Take now an arbitrary \( f \in X \). We claim that \( I - L \) is not surjective in \( W(\theta_\lambda, 0) \), showing that \((f, \psi)\) does not belong to \( Rg(I - L) \) in \( W(\theta_\lambda, 0) \). Indeed, assume that there exists \((\xi_1, \eta_1) \in W(\theta_\lambda, 0) \) such that \((I - L)(\xi_1, \eta_1)^t = (f, \psi)\). Taking the second equation
\[
\left(\Delta - \frac{\mu + c\theta_\lambda}{1 + \beta\theta_\lambda}\right) \eta_1 = \varphi,
\]
and multiplying by \( \eta \), we arrive at \( \int_\Omega \eta \varphi = 0 \), an absurdum.
In a similar way, it can be shown that \( i_W(K, (0, \Theta_{\mu,\alpha})) = 0 \).

Hence, by (I.1)-(I.3) we can deduce the existence of a coexistence state \((u, w)\) of (3.29) for \((\lambda, \mu)\) verifying (1.9) and \(\lambda > \lambda_1\) or \(\mu > \lambda_1\). But, observe that if \((\lambda, \mu)\) verifies (1.9), then \(\lambda > \lambda_1\) or \(\mu > \lambda_1\). This completes the proof.

\[ \square \]

6 Limiting systems and solutions when \(\alpha \to \infty\) and \(\beta \to \infty\).

We start with the limit problem when \(\beta \to \infty\). We will detail only (I) and (II), since (III) follows from the same arguments of the proof of Theorem 1.2 in [2].

Proof of Theorem 1.2. (I) Assume that \(\lambda > \lambda_1\) and let \((u_\beta, w_\beta)\) be a coexistence state of (3.29). Since

\[ -\Delta w_\beta = w_\beta f(u_\beta, w_\beta) (\mu - w_\beta f(u_\beta, w_\beta) + cu_\beta) \quad \text{in } \Omega, \]

if \(w_\beta > 0\) we conclude that

\[ 0 = \lambda_1 (-f(u_\beta, w_\beta)(\mu + cu_\beta) + w_\beta f^2(u_\beta, w_\beta)) \]

\[ > \lambda_1 (-f(u_\beta, w_\beta)(\mu + cu_\beta)) \to \lambda_1 > 0 \]

as \(\beta \to \infty\), because \(\theta_\lambda \leq u_\beta\) and then \(\beta \theta_\lambda \leq \beta u_\beta\). Hence, \(w_\beta \equiv 0\) and consequently \(u_\beta \equiv \theta_\lambda\).

(II) Denote by \(z_\beta = \beta u_\beta\) and \(w_\beta = (1 + \alpha v + z_\beta)v_\beta\). Then, the pair \((z_\beta, w_\beta)\) verifies the system

\[
\begin{align*}
-\Delta z_\beta &= z_\beta (\lambda - \frac{1}{\beta} z_\beta + b \frac{2w_\beta}{\sqrt{(1 + z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}}) \quad \text{in } \Omega, \\
-\Delta w_\beta &= \frac{2w_\beta}{\sqrt{(1 + z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}} (\mu - \frac{2w_\beta}{\sqrt{(1 + z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}} + \frac{c}{\beta} z_\beta) \quad \text{in } \Omega, \\
z_\beta &= w_\beta = 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (6.41)

Observe that

\[ \frac{c}{\beta} \frac{z_\beta}{\sqrt{(1 + z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}} \to 0 \quad \text{uniformly in } \Omega \text{ as } \beta \to \infty. \] (6.42)
We claim that
\[ \|w_\beta\|_{L^\infty(\Omega)} \leq C. \] (6.43)

Take \( \varepsilon > 0 \) such that \( \lambda_1(-\varepsilon) = \lambda_1 - \varepsilon > 0 \). For such \( \varepsilon > 0 \), using (6.42) there exists \( \beta_0 \) such that for \( \beta \geq \beta_0 \) we get
\[ (-\Delta - \varepsilon)w_\beta \leq \frac{2w_\beta}{\sqrt{(1+z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}} (\mu - \frac{2w_\beta}{\sqrt{(1+z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}}) \leq \frac{\mu^2}{4} \] in \( \Omega \),
\[ w_\beta = 0 \] on \( \partial\Omega \),
and then, we get that
\[ w_\beta \leq \hat{\xi}_{[\beta^2/4]}, \]
where now \( \hat{\xi}_{[\beta]} \) is the unique solution of
\[ (-\Delta - \varepsilon)\hat{\xi} = b(x) \text{ in } \Omega, \quad \hat{\xi} = 0 \text{ on } \partial\Omega, \]
for \( b \in L^\infty(\Omega) \). Hence, we conclude (6.43).

Now, assume that \( \|z_\beta\|_\infty \to \infty \) as \( \beta \to \infty \). Then
\[ \left\| \frac{2w_\beta}{\sqrt{(1+z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}} \right\|_\infty \to 0. \]
Hence
\[ 0 = \lambda_1 \left( -\lambda + \frac{1}{\beta}z_\beta - b \frac{2w_\beta}{\sqrt{(1+z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}} \right) > \lambda_1 \left( -\lambda - b \frac{2w_\beta}{\sqrt{(1+z_\beta)^2 + 4\alpha w_\beta + 1 + z_\beta}} \right) \to \lambda_1 - \lambda > 0, \]
and then \( z_\beta \equiv 0 \), a contradiction.

Thus, \( \|z_\beta\|_\infty \leq C \). Passing to the limit in (6.41), we conclude that \( (z_\beta, w_\beta) \to (z, p) \) in \((C^2(\overline{\Omega}))^2\) with
\[
\begin{cases}
-\Delta z = z(\lambda + b \frac{2p}{\sqrt{(1+z)^2 + 4\alpha p + 1 + z}}) & \text{in } \Omega, \\
-\Delta p = \frac{2p}{\sqrt{(1+z)^2 + 4\alpha p + 1 + z}} (\mu - \frac{2p}{\sqrt{(1+z)^2 + 4\alpha p + 1 + z}}) & \text{in } \Omega, \\
z = p = 0 & \text{on } \partial\Omega.
\end{cases}
\] (6.44)

After the change of variable \( p = (1 + \alpha w + z)w \), it is clear that (1.10) is equivalent to (6.44), and the result follows. \( \square \)
Next we analyze the profile of the solutions when $\alpha \to \infty$.

Proof of Theorem 1.3. (I) Assume that $\lambda < \lambda_1$. If $w \equiv 0$, then since $\lambda < \lambda_1$ we get $u \equiv 0$. Assume that $w > 0$. Observe that

$$-\Delta w \leq \frac{2w}{\sqrt{(1 + \beta u)^2 + 4\alpha w + (1 + \beta u)}} (\mu + cw_M) \leq C \left(\frac{w}{\alpha}\right)^{1/2},$$

for some $C > 0$, where we have used that $bc < 1$ and then $u$ is bounded. Then,

$$-\Delta (\alpha w) \leq C (\alpha w)^{1/2} \quad \text{in } \Omega,$$

thus we conclude that for some positive constant $C_1 > 0$

$$\alpha w \leq C_1 \quad \text{in } \Omega. \quad (6.45)$$

As consequence, $\|w\|_{L^\infty(\Omega)} \to 0$ as $\alpha \to \infty$. Since $\lambda < \lambda_1$ we can take $\varepsilon > 0$ such that $\lambda + \varepsilon < \lambda_1$. For such $\varepsilon > 0$, there exists $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$ we have

$$-\Delta u = u(\lambda - u + b \frac{2w}{\sqrt{(1 + \beta u)^2 + 4\alpha w + (1 + \beta u)}}) \leq u(\lambda - u + 2bw) \leq u(\lambda + \varepsilon - u).$$

Since $\lambda + \varepsilon < \lambda_1$, this implies that $u \equiv 0$.

(II) Denote by $z_\alpha = \alpha v_\alpha$ and

$$W_\alpha = (1 + z_\alpha + \beta u_\alpha)z_\alpha.$$

In this case, $(u_\alpha, W_\alpha)$ verifies

$$\begin{cases}
-\Delta u_\alpha = u_\alpha(\lambda - u_\alpha + \frac{b}{\alpha} \frac{2W_\alpha}{\sqrt{(1 + \beta u_\alpha)^2 + 4W_\alpha + (1 + \beta u_\alpha)}}) \quad \text{in } \Omega, \\
-\Delta W_\alpha = \frac{2W_\alpha}{\sqrt{(1 + \beta u_\alpha)^2 + 4W_\alpha + (1 + \beta u_\alpha)}} (\mu) \\
\quad \quad \quad \quad \quad - \frac{1}{\alpha} \frac{2W_\alpha}{\sqrt{(1 + \beta u_\alpha)^2 + 4W_\alpha + (1 + \beta u_\alpha)}} + cw_\alpha) \quad \text{in } \Omega, \\
u_\alpha = W_\alpha = 0 \quad \text{on } \partial \Omega.
\end{cases} \quad (6.46)$$

Observe that $W_\alpha = \alpha w_\alpha$ and then $W_\alpha$ is bounded by (6.45). Hence,

$$\frac{W_\alpha}{\sqrt{(1 + \beta u_\alpha)^2 + 4W_\alpha + (1 + \beta u_\alpha)}} \leq W_\alpha^{1/2} \leq C,$$

and then

$$\frac{b}{\alpha} \frac{2W_\alpha}{\sqrt{(1 + \beta u_\alpha)^2 + 4W_\alpha + (1 + \beta u_\alpha)}} \to 0 \quad \text{uniformly in } \Omega \text{ as } \alpha \to \infty.$$
So, using elliptic regularity, we obtain that
\[ u_\alpha \to \theta_\lambda \text{ in } C^2(\Omega) \text{ as } \alpha \to \infty. \]

Coming back to the equation of \( W_\alpha \) we conclude that \( W_\alpha \to W^* \) in \( C^2(\Omega) \) with
\[ -\Delta W^* = \frac{2W^*}{\sqrt{(1 + \beta \theta_\lambda)^2 + 4W^* + (1 + \beta \theta_\lambda)}} (\mu + c\theta_\lambda). \]

This concludes the result.

(III) This paragraph follows by Proposition 2.3.

\[ \square \]

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**References**


A symbiotic self-cross diffusion model


