Some topics about Nematic and Smectic-A Liquid Crystals

Chillán, enero de 2010
<table>
<thead>
<tr>
<th></th>
<th>Introduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>The Models</td>
</tr>
<tr>
<td>3</td>
<td>Statement of the Problems</td>
</tr>
<tr>
<td>4</td>
<td>Nematic Case</td>
</tr>
<tr>
<td>5</td>
<td>Smectic-A Case</td>
</tr>
<tr>
<td></td>
<td>Section</td>
</tr>
<tr>
<td>---</td>
<td>------------------</td>
</tr>
<tr>
<td>1</td>
<td>Introduction</td>
</tr>
<tr>
<td>2</td>
<td>The Models</td>
</tr>
<tr>
<td>3</td>
<td>Statement of the Problems</td>
</tr>
<tr>
<td>4</td>
<td>Nematic Case</td>
</tr>
<tr>
<td>5</td>
<td>Smectic-A Case</td>
</tr>
</tbody>
</table>
1. Introduction

2. The Models

3. Statement of the Problems

4. Nematic Case

5. Smectic-A Case
Table of contents

1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case
thermotropic liquid crystals

3-D lattice
orientation
solid

1- (2-)D lattice
orientation
fluid

no lattice
orientation
fluid

no lattice
no orientation
fluid

anisotropic
anisotropic
anisotropic
isotropic

http://moebius.physik.tu-berlin.de/lc/lcs.html
Introduction

http://www.doitpoms.ac.uk/
Introduction

Natural examples:
- Soap, soup
- Biological membranes
- The protein solution to generate silk of a spider
- DNA and polypeptides can form LC phases

Applications:
- Liquid Crystal Displays: wrist watches, pocket calculators, flat screens ...
- Liquid Crystal Thermometers: to show a map of temperatures to find tumors, bad connections on a circuit board ...
- Windows that can be changed from clear and opaque with the flip of a switch.
- To make a stable hydrocarbon foam.
- Optical Imaging and recording.
Introduction

Natural examples:
- Soap, soup
- Biological membranes
- The protein solution to generate silk of a spider
- DNA and polypeptides can form LC phases

Applications:
- Liquid Crystal Displays: wrist watches, pocket calculators, flat screens ...
- Liquid Crystal Thermometers: to show a map of temperatures to find tumors, bad connections on a circuit board ...
- Windows that can be changed from clear and opaque with the flip of a switch.
- To make a stable hydrocarbon foam.
- Optical Imaging and recording.
http://atom.physics.calpoly.edu
http://en.wikipedia.org/
Table of contents

1 Introduction

2 The Models

3 Statement of the Problems

4 Nematic Case

5 Smectic-A Case
**The models**

- **\( \mathbf{d} \)**: Orientation of liquid crystal molecules (unit vector).
- **\( \mathbf{n} \)**: Single optical axis perpendicular to the layer.

\[ |\mathbf{d}| = 1 \quad \rightarrow \quad f \text{ Ginzburg-Landau penalization function} \]

\[ f(\mathbf{d}) = \frac{1}{\varepsilon^2} (|\mathbf{d}|^2 - 1) \mathbf{d} \]

- \( \nabla \times \mathbf{n} = 0 \quad \rightarrow \quad \mathbf{n} = \nabla \varphi \)

\( \varphi \): Layer variable

- \( \mathbf{d} = \mathbf{n} \quad \rightarrow \quad |\nabla \varphi| = 1 \)
The models

Penalized Oseen–Frank energy:

\[(\text{NC}) \quad E_e = \int_\Omega \left( \frac{1}{2} |\nabla d|^2 + F(\varphi) \right) \quad (\text{SAC}) \quad E_e = \int_\Omega \left( \frac{1}{2} |\Delta \varphi|^2 + F(\nabla \varphi) \right)\]

where \(f(n) = \nabla_n F(n)\).

\(F(n) = \frac{1}{4\varepsilon^2} (|n|^2 - 1)^2\) potential function of \(f(n) = \frac{1}{\varepsilon^2} (|n|^2 - 1)n\).

Minimization problem \(\longrightarrow\) Euler-Lagrange equation

\[(\text{NC}) \quad \omega \equiv \Delta d - f(d) = 0, \quad (\text{SAC}) \quad \omega \equiv \Delta^2 \varphi - \nabla \cdot f(\nabla \varphi) = 0,\]
Penalized Oseen–Frank energy:

\[(\text{NC}) \quad E_e = \int_{\Omega} \left( \frac{1}{2} |\nabla d|^2 + F(\varphi) \right) \quad \text{(SAC)} \quad E_e = \int_{\Omega} \left( \frac{1}{2} |\Delta \varphi|^2 + F(\nabla \varphi) \right)\]

where \( f(n) = \nabla_n F(n) \).

\[F(n) = \frac{1}{4\varepsilon^2} (|n|^2 - 1)^2 \]

potential function of \( f(n) = \frac{1}{\varepsilon^2} (|n|^2 - 1)n \).

Minimization problem \(\rightarrow\) Euler-Lagrange equation

\[(\text{NC}) \quad \omega \equiv \Delta d - f(d) = 0, \quad \text{(SAC)} \quad \omega \equiv \Delta^2 \varphi - \nabla \cdot f(\nabla \varphi) = 0,\]
The models

\( \Omega \subset \mathbb{R}^N \ (N = 2 \text{ or } 3), \ \partial \Omega \text{ regular, } Q = \Omega \times (0, +\infty) \)

(Ericksen-Leslie, Lin, E)

Angular momentum

\begin{align*}
(\text{NC}) \quad \partial_t d + u \cdot \nabla d + \gamma \omega &= 0 \\
(\text{SAC}) \quad \partial_t \phi + u \cdot \nabla \phi + \gamma \omega &= 0
\end{align*}

Linear momentum

\[ \rho \left( \partial_t u + (u \cdot \nabla) u \right) - \nabla \cdot (\sigma^d + \lambda \sigma^e) + \nabla p = 0, \quad \nabla \cdot u = 0 \]

where
The models

\[ \Omega \subset \mathbb{R}^N (N = 2 \text{ or } 3), \partial \Omega \text{ regular}, \ Q = \Omega \times (0, +\infty) \]

( Ericksen-Leslie, Lin, E)

Angular momentum

\[ \partial_t d + u \cdot \nabla d + \gamma \omega = 0 \]  

\[ \partial_t \varphi + u \cdot \nabla \varphi + \gamma \omega = 0 \]  

Linear momentum

\[ \rho (\partial_t u + (u \cdot \nabla) u) - \nabla \cdot (\sigma^d + \lambda \sigma^e) + \nabla p = 0, \ \nabla \cdot u = 0 \]

where
The models

(NC)

\[ \sigma^d = \mu_4 D(u), \]
\[ \sigma^e = \lambda \nabla \cdot (\nabla d \otimes \nabla d) \]

(SAC)

\[ \sigma^d = \mu_1 (n^t D(u)n) n \otimes n + \mu_4 D(u) + \mu_5 (D(u)n \otimes n + n \otimes D(u)n), \]
\[ \sigma^e = -f(n) \otimes n + \nabla(\nabla \cdot n) \otimes n - (\nabla \cdot n) \nabla n \]
1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case
The equations

\[
\begin{align*}
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= -\nabla \mathbf{d}^t \Delta \mathbf{d}, \\
\nabla \cdot \mathbf{u} &= 0, \\
\partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla)\mathbf{d} &= (\Delta \mathbf{d} - f(\mathbf{d})), \quad |\mathbf{d}| \leq 1,
\end{align*}
\]

in \( Q \)

+ time-dependent (bc) on \( \Sigma = (0, \infty) \times \partial \Omega \).

+ (iv) or (tp) in \( \Omega \).
Smectic Model

The equations

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u - \nabla \cdot \sigma_{nl}^d \\
-(\Delta^2 \varphi - \nabla \cdot f(\nabla \varphi)) \nabla \varphi + \nabla q &= 0, \\
\nabla \cdot u &= 0, \\
\partial_t \varphi + u \cdot \nabla \varphi + (\Delta^2 \varphi - \nabla \cdot f(\nabla \varphi)) &= 0,
\end{align*}
\]

in \( Q \)

where \( \sigma_{nl}^d := \langle n^t D(u)n \rangle n \otimes n + D(u)n \otimes n + n \otimes D(u)n \)

+ time-dependent (bc) on \( \Sigma = (0, \infty) \times \partial \Omega \).
+ (iv) or (tp) in \( \Omega \).
Lifting functions

Boundary condition depending on the time $\rightsquigarrow$

\[(\text{NC}) \quad \tilde{d} = \tilde{d}(t)\]

stationary (for weak norms) or non-stationary (for regular norms) lifting function

\[\hat{d}(t) = d(t) - \tilde{d}(t), \quad \tilde{d} = 0 \text{ on } \partial \Omega\]

Unknows: $u, p, \hat{d}$

\[(\text{SAC}) \quad \tilde{\varphi} = \tilde{\varphi}(t)\]

non stationary lifting function

\[\hat{\varphi}(t) = \varphi(t) - \tilde{\varphi}(t), \quad \tilde{\varphi} = 0 \text{ on } \partial \Omega\]

Unknows: $u, p, \hat{\varphi}$
1. Introduction

2. The Models

3. Statement of the Problems

4. Nematic Case

5. Smectic-A Case
Nematic Case

Weak solution:

\[ u \in L^\infty(0, +\infty; L^2) \cap L^2_w(0, +\infty; H^1), \]
\[ d \in L^\infty(0, +\infty; H^1) \cap L^2_w(0, +\infty; H^2) \]

Regular solution:

\[ u \in L^\infty(0, +\infty; H^1) \cap L^2_w(0, +\infty; H^2), \]
\[ d \in L^\infty(0, +\infty; H^2) \cap L^2_w(0, +\infty; H^3) \]
Asymptotic Stability:

\[ E(t) = \frac{1}{2} |u(t)|^2 + E_e(t) \rightarrow E_\infty, \]

\[ u(t) \rightarrow 0 \text{ in } H^1_0(\Omega), \quad \omega(t) = (\Delta \varphi - f(\varphi))(t) \rightarrow 0 \text{ in } L^2(\Omega) \]

when \( t \uparrow +\infty \), where \( E_\infty = E_{e,d} = \frac{1}{2} |\nabla \overline{d}|^2 + \int_\Omega F(\overline{d}) \) and \( \overline{d} \) is a critical point of \( E_e \). Moreover, \( d \rightarrow \overline{d} \) ”for subsequences” in \( H^2(\Omega) \)-weak.

Stability:

If initial data are small, \( |u|, |\omega|, \) and \( E(t) \) are small for each \( t \geq 0 \).
**Previous result:** (iv)-problem, boundary condition independent of time. Existence of a weak global solution. Existence and uniqueness of a regular solution for larger viscosity. [Lin,Liu’95]

**Goal**

Time-dependent (bc) case

- [Climent, Guillén, Rojas’06]

- [Climent, Guillén, Moreno’08]
  (tp)-problem. Existence of a regular solution, $\nu$ big enough.

- [Climent, Guillén, Rodríguez] Stability and asymptotic stability (tp)-problem, time-independent (bc) case

Goal Time-dependent (bc) case


- [Climent,Guillén,Rodríguez] Stability and asymptotic stability (tp)-problem, time-independent (bc) case
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
</tr>
<tr>
<td>2</td>
<td>The Models</td>
</tr>
<tr>
<td>3</td>
<td>Statement of the Problems</td>
</tr>
<tr>
<td>4</td>
<td>Nematic Case</td>
</tr>
<tr>
<td>5</td>
<td>Smectic-A Case</td>
</tr>
</tbody>
</table>
Smectic-A Case

Weak solution:

\[ u \in L^\infty(0, +\infty; H^1) \cap L^2_w(0, +\infty; H^2), \]
\[ \varphi \in L^\infty(0, +\infty; H^4) \cap L^2_w(0, +\infty; H^6) \]

Regular solution:

\[ u \in L^\infty(0, +\infty; H^1) \cap L^2_w(0, +\infty; H^2), \]
\[ \varphi \in L^\infty(0, +\infty; H^2) \cap L^2_w(0, +\infty; H^3) \]
Smectic-A Case

Asymptotic Stability:

\[ u(t) \to 0 \text{ in } H^1_0(\Omega) \quad (\Delta^2 \phi - \nabla \cdot f(\nabla \phi))(t) \to 0 \text{ in } L^2(\Omega) \]

when \( t \uparrow +\infty \). Moreover, \( \phi \to \overline{\phi} \) for ”sequences” in \( H^4(\Omega) \)-weak, where \( \overline{\phi} \) is a solution of a Euler-Lagrange problem.

Stability:

If initial data are small, \( u, \phi, \) and \( \omega \) are small for each \( t \geq 0 \).
**Previous result:** (iv)-Problem, time-independent boundary conditions. Existence of weak solutions in $[0, T]$, global regularity of weak solutions (for big enough viscosity). [Liu’00]

**Goal** Time-dependent (bc) case [Climent, Guillén]

1. Uniqueness weak/strong solutions (iv)-Problem,
2. Existence of global weak solutions (iv)-Problem, “bounded” up to infinity time,
3. Existence of weak time-periodic solutions,
4. Existence of regular solutions for both previous cases (dominant viscosity coefficient).
5. Stability and asymptotic stability Time-independent (bc) case
Previous result: (iv)-Problem, time-independent boundary conditions. Existence of weak solutions in $[0, T]$, global regularity of weak solutions (for big enough viscosity). [Liu’00]

Goal Time-dependent (bc) case [Climent, Guillén]

1. Uniqueness weak/strong solutions (iv)-Problem,
2. Existence of global weak solutions (iv)-Problem, “bounded” up to infinity time,
3. Existence of weak time-periodic solutions,
4. Existence of regular solutions for both previous cases (dominant viscosity coefficient).
5. Stability and asymptotic stability Time-independent (bc) case