Hilbert's Programme and Ordinal Analysis

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Hilbert’s second problem in his famous 1900 list of mathematical problems was the “consistency of the axioms of Analysis”.

Towards a solution of his second problem he later (around 1922 in [7]) launched a program that today is known as Hilbert’s programme.

However, by Gödel’s incompleteness theorems (around 1931 in [6]) the main aim of Hilbert’s programme, a rigid consistency proof by intangible finitistic means, turned out to be impossible.
Hilbert’s second problem in his famous 1900 list of mathematical problems was the “consistency of the axioms of Analysis”.
Towards a solution of his second problem he later (around 1922 in [7]) launched a program that today is known as Hilbert’s programme.
However, by Gödel’s incompleteness theorems (around 1931 in [6]) the main aim of Hilbert’s programme, a rigid consistency proof by intangible finitistic means, turned out to be impossible. So Hilbert’s programme failed in this aspect.
Unperturbed by Gödel’s results Gentzen around 1936 (in [3]) published his first version of a consistency proof for Peano arithmetic and “improved” it in [4]. This paper and its successors [4] and especially [5] gave birth to a branch of proof theory which we today call ordinal analysis. Clearly Gentzen’s papers (at least the first two) aimed at the consistency aspect of Hilbert’s programme. From a rigid finitist point of view Gentzen’s efforts failed in this aspect.
Unperturbed by Gödel’s results Gentzen around 1936 (in [3]) published his first version of a consistency proof for Peano arithmetic and “improved” it in [4]. This paper and its successors [4] and especially [5] gave birth to a branch of proof theory which we today call ordinal analysis. Clearly Gentzen’s papers (at least the first two) aimed at the consistency aspect of Hilbert’s programme. From a rigid finitist point of view Gentzen’s efforts failed in this aspect. However, in this talk I want to indicate that ordinal analysis contributes to another aspect of Hilbert’s programme, the elimination of ideal objects.
This is a passage of his 1927 talk given in Hamburg

“Der Physiker verlangt gerade von einer Theorie, daß ohne die Heranziehung von anderweitiger Bedingungen aus den Naturgesetzen oder Hypothesen die besonderen Sätze allein durch Schlüsse, also auf Grund eines reinen Formelspiels abgeleitet werden. Nur gewisse Kombinationen und Folgerungen der physikalischen Gesetze können durch Experimente kontrolliert werden — so wie in meiner Beweistheorie nur die realen Aussagen unmittelbar einer Verifikation fähig sind.”
Which in my translation says:

*The physicist requires for a theory that its theorems can be formally derived from the laws of nature and its hypotheses alone without referring to outside perceptions. Only certain combinations and conclusions of physical laws are checkable by experiments — this is also true for my proof theory in which only “real statements” are verifiable.*
Which statements are verifiable?

- All instances of a $\Pi^0_1$-sentence.
- All instances of $\Pi^0_2$-sentences $(\forall x)(\exists y)F(x, y)$ provided that there is a testing function for it, i.e. a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $m \in \mathbb{N}$ there is a $k \leq f(m)$ such that $F(m, k)$ is true in $\mathbb{N}$. Therefore we may say that in a rigid sense only $\Pi^0_2$-statements are “real” while more complex statements have already to be regarded as “ideal”.
Which statements are verifiable?

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Therefore we may say that in a rigid sense only $\Pi^0_2$–statements are “real” while more complex statements have already to be regarded as “ideal”.
If a $\Pi^0_2$–statement $(\forall x)(\exists y)\phi(x, y)$ is provable in a theory $T$ the function $x \mapsto \min \{y \mid \phi(x, y)\}$ is computable, hence a testing function. Call it the testing function.

In analogy to the situation in physics we define:

**Definition**

A theory $T$ is *experimentally checkable* if there is a computable function $F_T$ that eventually majorizes all the testing functions for in $T$ provable $\Pi^0_2$ statements.

We say that $F_T$ designs an experiment for $T$. 
Definition ($\Pi^0_2$–analysis of an axiom system $T$)

Extract by elimination of the “ideal means” in $T$–proofs of $\Pi^0_2$–statements a computable function $F_T$ which designs an experiment for $T$. Clearly $F_T$ should be obtainable without reference to ideal means.
By an ordinal analysis of a mathematical theory $T$ we commonly understand the computation of its *proof theoretic ordinal*, i.e., the supremum of the order–types of in $T$ elementarily definable well–ordering whose well–foundedness is provable from the axioms of $T$. Though I do not want to become too technical I will sketch the main lines of an ordinal analysis. To do this we have to distinguish two forms of proof theory. *Predicative proof theory* which analyzes predicative theories and centers around subsystems of second order number theory and *impredicative proof theory* which is needed in the analysis of impredicative theories and today centers primarily around subsystems of set theory.
The biggest progress since Gentzen is the use of infinitary systems in the ordinal analysis of predicative theories. The standard example is the use of $\omega$–logic in the ordinal analysis of Peano arithmetic. In the language of second order number theory well–foundedness is expressed by a $\Pi^1_1$–sentence which in turn in first order number theory can be expressed by a formula with free set variables.
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**Theorem ($\omega$–completeness)**

A $\Pi^1_1$–sentence $(\forall X)F(X)$ is true in the standard structure $\mathbb{N}$ if there is a cut–free proof of $F(X)$ in first order $\omega$–logic.

**Definition (Truth complexity)**

The truth complexity $tc((\forall X)F(X))$ of a $\Pi^1_1$–sentence $(\forall X)F(X)$ is the minimum of the order–types of cut free proof trees for $F(X)$ in first order $\omega$–logic.
Theorem (Boundedness)

Let \( \prec \) be a well-ordering on the natural numbers of limit order type. Then the order-type of \( \prec \) is equal to the truth complexity of the sentence

\[
(\forall X)[(\forall x)[(\forall y \prec x)y \in X \rightarrow x \in X] \rightarrow (\forall x)[x \in X]].
\]

Hence

\[
|T| \leq \sup \{\text{tc}(F) \mid T \vdash F}\]

where \( |T| \) means the proof theoretic ordinal of the axiom system \( T \).
The standard procedure for ordinal analysis of predicative theories.

\[ T \vdash F \Rightarrow \exists \alpha, \rho \, |_{\rho}^{\alpha} F \Rightarrow \exists f \, |_{0}^{f(\alpha, \rho)} F. \]

Hence \(|T| \leq \min \{ \pi \mid \alpha < \pi \land \rho < \pi \land \pi \text{ is closed under } f \}\).
The standard procedure for ordinal analysis of predicative theories.

\[ T \vdash F \implies \exists \alpha, \rho \left\| \frac{\alpha}{\rho} F \implies \exists f \left\| \frac{f(\alpha, \rho)}{0} F \right. \right. \]

Hence \( |T| \leq \min \{ \pi | \alpha < \pi \land \rho < \pi \land \pi \text{ is closed under } f \} \).

For predicative systems the function \( f \) is essentially the Veblen function. More precisely we have \( f(\alpha, \rho) < \varphi_\sigma(0) \) for \( \rho < \omega^\sigma \) and \( \alpha < \varphi_\sigma(0) \).

This yields the Schütte–Feferman ordinal

\[ \Gamma_0 := \min \{ \alpha | \varphi_\alpha(0) = \alpha \} \]

— the least ordinal that is closed under \( \varphi \) viewed as a binary function — as an upper bound for the order–type of predicative systems.

Since all ordinals less than \( \Gamma_0 \) are predicatively attainable, \( \Gamma_0 \) is regarded as the bounding ordinal for predicativity.
Let $\mathcal{S}$ be a structure and $\mathcal{L}_\mathcal{S}$ its language. We divide the sentences of $\mathcal{L}_\mathcal{S}$ into two types, $\wedge$–type and $\vee$–type, and decorate each sentence $F$ with a characteristic sequence $\text{CS}(F)$ such that

- $\mathcal{S} \models F$ iff $\mathcal{S} \models G$ for all $G \in \text{CS}(F)$ for $F \in \wedge$–type
- $\mathcal{S} \models F$ iff $\mathcal{S} \models G$ for some $G \in \text{CS}(F)$ for $F \in \vee$–type
Let $\mathcal{G}$ be a structure and $\mathcal{L}_\mathcal{G}$ its language. We divide the sentences of $\mathcal{L}_\mathcal{G}$ into two types, $\land$–type and $\lor$–type, and decorate each sentence $F$ with a characteristic sequence $\text{CS}(F)$ such that

- $\mathcal{G} \models F$ iff $\mathcal{G} \models G$ for all $G \in \text{CS}(F)$ for $F \in \land$–type
- $\mathcal{G} \models F$ iff $\mathcal{G} \models G$ for some $G \in \text{CS}(F)$ for $F \in \lor$–type

The verification calculus for $\mathcal{G}$ is given by

$$(\land) \quad F \in \land\text{–type}, \quad \frac{\Delta, G \text{ and } \alpha_G < \alpha \text{ for all } G \in \text{CS}(F)}{\frac{\alpha}{\Delta, F}}$$

$$(\lor) \quad F \in \lor\text{–type}, \quad \frac{\Delta, G \text{ and } \alpha_0 < \alpha \text{ for some } G \in \text{CS}(F)}{\frac{\alpha}{\Delta, F}}$$
By adding additional rules, among them the cut rule

\[(\text{cut}) \quad \text{If } \frac{\alpha}{\rho} \Delta, F, \frac{\alpha}{\rho} \Delta, \neg F, \ \text{rnk}(F) < \rho \text{ and } \alpha < \beta \text{ then } \frac{\beta}{\rho} \Delta,\]

we get a *semi–formal* system \(\frac{\alpha}{\rho} \Delta\).

**Theorem (Semantical cut elimination)**

Let \(\frac{\alpha}{\rho} \Delta\) be an \(\mathcal{G}\)–sound semi–formal system that only derives sentences. Then \(\frac{\alpha}{\rho} \Delta\) already implies \(\frac{\alpha}{\rho} \Delta\).
For impredicative theories we stick to the example of theories in the language of set theory. We only regard theories that have models in the constructible hierarchy.

**Definition**

Let $T$ be an axiom system in the language of set theory that possesses a model in the constructible hierarchy. For an admissible ordinal $\pi > \omega$ let

$$\| T \|_\pi := \min \{ \alpha \mid L_\alpha \models F \text{ and } T \vdash F^{L_\pi} \},$$

where $F$ is supposed to be a $\Pi_2$–formula. Clearly this definition requires that $T$ can talk about $L_\pi$. 
For theories $T$ in the language of set theory which comprise KP$_\omega$ we have $|T| = \|T\|^{\omega_1}_{CK}$ where $|T|$ stands for its proof theoretic ordinal.

Observe that the definition of the ordinal $\|T\|^{\omega_1}_{CK}$ does not need second order variables.

In the ordinal analysis of $T$ we may thus use a semi–formal system for its standard structure — the constructible hierarchy.
To fix the language for a verification calculus for L we define the language of *ramified set theory*. The terms and sentences of $\mathcal{L}_{RS}$ are given by the following clauses:

- For every ordinal $\alpha$ the term $L_\alpha$ is an atomic term of stage $\alpha$ and
- if $a_1, \ldots, a_n$ are terms of stages less than $\alpha$ and $F(x, x_1, \ldots, x_n)$ is a formula in the language of set theory then $\{x \in L_\alpha \mid F(x, a_1, \ldots, a_n)^{L_\alpha}\}$ is a composed term of stage $\alpha$.
- If $F(x_1, \ldots, x_n)$ is a $\Delta_0$–formula in the language of set theory whose free variables occur all in the list $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ are terms of ramified set theory then $F(a_1, \ldots, a_n)$ is a sentence of ramified set theory.

By $L_{\lessdot \alpha}$ we denote the sequence of all $\mathcal{L}_{RS}$–terms of stages less than $\alpha$ ordered by the relation $\prec_L$. 
The $\lor$–type of $\mathcal{L}_{RS}$ comprises all sentences of the form $a \in b$, $F \lor G$ and $(\exists x \in a)F(x)$ and dually the $\land$–type comprises the sentences $a \notin b$, $F \land G$ and $(\forall x \in a)F(x)$.

For the sentences in $\lor$–type the decoration is given by

- $\text{CS}(a \in L_\alpha) := \langle a = s \mid s \in L_{<\alpha} \rangle$,
- $\text{CS}(a \in \{x \in L_\alpha \mid F(x)\}) := \langle a = s \land F(s) \mid s \in L_{<\alpha} \rangle$,
- $\text{CS}(F \lor G) := \langle F, G \rangle$,
- $\text{CS}((\exists x \in L_\alpha)F(x)) := \langle F(s) \mid s \in L_{<\alpha} \rangle$,
- $\text{CS}((\exists x \in a)F(x)) := \langle F(s) \land G(s) \mid s \in L_{<\alpha} \rangle$ if $a$ is a composed term $\{x \in L_\alpha \mid G(x)\}$.

Having defined a decoration for $\mathcal{L}_{RS}$ we obtain a verification calculus $\models^\alpha \Delta$ for finite sets of $\mathcal{L}_{RS}$–sentences.
There is also a counterpart of the boundedness theorem which roughly says that $\models_\alpha F^{L_\beta}$ and $\alpha \leq \beta$ for a $\Sigma$–sentence $F$ containing only parameters from $L_{<\alpha}$ entails $L_\alpha \models F$. 
There is also a counterpart of the boundedness theorem which roughly says that $\models^\alpha F^L$ and $\alpha \leq \beta$ for a $\Sigma$–sentence $F$ containing only parameters from $L_{<\alpha}$ entails $L_\alpha \models F$.

The predicative standard procedure for an ordinal analysis would this yield

$$T \models F^{L_\omega^1CK} \Rightarrow \exists \alpha \exists \rho \models^\alpha F^{L_\omega^1CK} \Rightarrow \models^\alpha F^{L_\omega^1CK} \Rightarrow L_\alpha \models F$$

for $\Pi_2$–sentences.
There is also a counterpart of the boundedness theorem which roughly says that \( \models F^L_{\beta} \) and \( \alpha \leq \beta \) for a \( \Sigma \)–sentence \( F \) containing only parameters from \( L_{<\alpha} \) entails \( L_{\alpha} \models F \).

The predicative standard procedure for an ordinal analysis would this yield

\[
T \vdash F^L_{\omega_1^{CK}} \Rightarrow \exists \alpha \exists \rho \models F^L_{\omega_1^{CK}} \Rightarrow \models F^L_{\omega_1^{CK}} \Rightarrow L_{\alpha} \models F
\]

for \( \Pi_2 \)–sentences. Which is apparently too simple.
Extending the verification calculus by a cut suffices for ordinal analyses of predicative systems but is too weak for impredicative ones. Here we need “ideal” ordinals which have to be axiomatized by a rule.
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An example is the ordinal $\Omega$ which can be axiomatized by a rule

$$
\frac{\alpha \rho}{\Delta, F} \implies \frac{\beta \rho}{\Delta, (\exists a \in L_\Omega)[a \neq \emptyset \land Tran(a) \land F^a]}
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for $\Pi_2$–formulas $F$. 
Extending the verification calculus by a cut suffices for ordinal analyses of predicative systems but is too weak for impredicative ones. Here we need “ideal” ordinals which have to be axiomatized by a rule. An example is the ordinal Ω which can be axiomatized by a rule

$$\frac{\alpha}{\rho} \Delta, F^L_{\Omega} \Rightarrow \frac{\beta}{\rho} \Delta, (\exists a \in L_{\Omega})[a \neq \emptyset \land Tran(a) \land F^a]$$

for $\Pi_2$–formulas $F$.

To make the so obtained semi–formal system $L$–sound the least possible interpretation for $\Omega$ is $\omega^\text{CK}_1$. 
However, if $T$ contains the foundation scheme the first interpretation step

$$T \vdash F^L\omega_1^{CK} \Rightarrow (\exists \alpha)(\exists \rho) \mid_\rho F^L\omega$$

already requires ordinals $\alpha \geq \Omega$, i.e., $\alpha \geq \omega_1^{CK}$, which makes semantical cut elimination useless for an ordinal analysis.
However, if $T$ contains the foundation scheme the first interpretation step

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already requires ordinals $\alpha \geq \Omega$, i.e., $\alpha \geq \omega_1^{CK}$, which makes semantical cut elimination useless for an ordinal analysis. Therefore we need a collapsing function which collapses ordinals above $\Omega$ to ordinals below $\Omega$, i.e., in the assignment of ordinals to semi–formal derivations we must only use ordinals from a subset of the ordinals that contains gaps which are large enough to allow for collapsing.
An idea going back to Buchholz ([1]) is to use operator controlled derivations. An operator is just a mapping that maps sets of ordinals to sets of ordinals.

**Definition**

Let $\frac{\alpha}{\rho} \Delta$ be a derivation in a semi–formal system. We say that an operator $\mathcal{H}$ controls $\frac{\alpha}{\rho} \Delta$ — written as $\mathcal{H} \frac{\alpha}{\rho} \Delta$ — if $\alpha \in \mathcal{H}(\text{par}(\Delta))$ and for every inference

$$\frac{\alpha_i}{\rho} \Delta_i \Rightarrow \frac{\alpha}{\rho} \Delta$$

which is not an inference according to $(\wedge)$, we have $\text{par}(\Delta_i) \subseteq \mathcal{H}(\text{par}(\Delta))$.

Here $\text{par}(\Delta)$ is just the finite set of stages of terms occurring in $\Delta$. 

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\]

which is not an inference according to \((\land)\), we have \( par(\Delta_i) \subseteq \mathcal{H}(par(\Delta)) \).

Here \( par(\Delta) \) is just the finite set of stages of terms occurring in \( \Delta \).

The predicative cut elimination theorem remains true also for operator controlled semi–formal derivations.
Let $\mathcal{F}$ be a set of functions that assign ordinals to tuples of ordinals. The Skolem hull operator $\mathcal{H}^\mathcal{F}: \text{Pow}(On) \rightarrow \text{Pow}(On)$ generated by $\mathcal{F}$ is inductively defined by the clauses

- $\mathcal{H}^\mathcal{F}_0(X) = X$,
- $\mathcal{H}^\mathcal{F}_{n+1}(X) = \mathcal{H}^\mathcal{F}_n(X) \cup \{ f(\xi_1, \ldots, \xi_n) | \xi_1, \ldots, \xi_n \in \mathcal{H}^\mathcal{F}_n(X) \land f \in \mathcal{F} \}$.
- $\mathcal{H}^\mathcal{F}(X) = \bigcup_n \mathcal{H}^\mathcal{F}_n(X)$.

A Skolem hull operator which contains $+$ and $\lambda^\xi.\omega^\xi$ among its generators is called Cantorian closed.

Generated Skolem hull operators are apparently inflationary, monotone and idempotent and thus non iterable.
To obtain iterations we have to increase the set of generators in every iteration step. An ordinal $\xi$ is $\mathcal{H}$–inaccessible if $\xi \notin \mathcal{H}(\xi)$ — where $\xi$ is viewed as the set of its predecessors. So we define simultaneously iterations $\mathcal{H}^{\tilde{\delta}, \alpha}$ and functions $\psi^{\tilde{\delta}}$ such that

$$\psi^{\tilde{\delta}}(\alpha) := \min \{ \xi \mid \xi \notin \mathcal{H}^{\tilde{\delta}, \alpha}(\xi) \} = \min \{ \xi \mid \mathcal{H}^{\tilde{\delta}, \alpha}(\xi) = \xi \}$$

(1)

and augment the generators of $\mathcal{H}^{\tilde{\delta}, \alpha}$ by $\psi^{\tilde{\delta}} \upharpoonright \alpha$. 

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and augment the generators of $\mathcal{H}^{\tilde{\mathcal{G}},\alpha}$ by $\psi^{\tilde{\mathcal{G}}} \upharpoonright \alpha$.

Starting with $\mathcal{G} = \{+\}$ this would suffice for an ordinal analysis of Peano arithmetic.
To obtain Skolem hull operators which suffice for impredicative systems we have to add “ideal” ordinals also to their generators, e.g., include the function $\alpha \mapsto \alpha^+$ in the set of generators. Then $\mathcal{H}^\delta(X)$ is not longer transitive, even for transitive $X$. To obtain iterations of the so generated operator we simultaneously define the iterations $\mathcal{H}^\alpha$ and functions $\psi_{\kappa^+}$ by

- $\psi_{\kappa^+}(\alpha) := \min \{ \xi \mid \mathcal{H}^\alpha(\xi) \cap \kappa^+ = \xi \}$ and
- add all functions $\psi_{\kappa^+} \upharpoonright \alpha$ to the generators of $\mathcal{H}^\alpha$.

These operators sufficed for the analyses of iterated inductive definitions or, equivalently set theories which axiomatize iterated admissibility. Adding also a weakly inaccessible ordinal sufficed for the theory KPi, adding a weakly Mahlo cardinal suffices for the theory KPM.
Even stronger axiom systems, e.g., systems which comprise
$\Pi_3$–reflection, do not only require the addition of even larger
abstract ordinals but also a refinement in the iteration of the
Skolem hull operators. The iteration has to be combined with
an iteration of a thinning procedure. An example for a thinning
procedure is the Mahlo–operation
$$\mathcal{M}(X) := \{ \xi \in X \mid X \text{ is stationary in } \pi \}. $$
Starting with a (large) cardinal $\Xi$ we get an iteration of the Mahlo–operation
by defining $\mathcal{M}^0 = \mathcal{M}(\Xi)$, $\mathcal{M}^{\alpha+1} = \mathcal{M}(\mathcal{M}^\alpha)$ and taking the
diagonal intersection $\mathcal{M}^\lambda = \{ \xi \in \Xi \mid (\forall \eta < \xi) [\xi \in \mathcal{M}(\mathcal{M}^\eta)] \}$ at
limit levels $\lambda$. 
To combine this with the iterations of a Skolem hull operator $H$ we define the sets

\[ M^\alpha_\kappa := \{ \xi \in \kappa \mid (\forall \eta \in H^\alpha (\xi) \cap \alpha)[\xi \in M(M^\eta_\kappa)] \land H^\alpha (\xi) \cap \kappa = \xi \} \]

for regular cardinals $\kappa$, put $\Psi^\alpha_\kappa := \min M^\alpha_\kappa$ and let $H^\alpha$ be the Skolem hull operator whose generators are augmented by all functions $\Psi^\kappa_\kappa \upharpoonright \alpha$. Since $\Psi^\kappa_\kappa (\alpha) < \kappa$ these functions are collapsing.
To combine this with the iterations of a Skolem hull operator $\mathcal{H}$ we define the sets

$$M^\alpha_\kappa := \{ \xi \in \kappa | (\forall \eta \in \mathcal{H}^\alpha(\xi) \cap \alpha)[\xi \in M(M^\eta_\kappa)] \land \mathcal{H}^\alpha(\xi) \cap \kappa = \xi \}$$

for regular cardinals $\kappa$, put $\Psi^\alpha_\kappa := \min M^\alpha_\kappa$ and let $\mathcal{H}^\alpha$ be the Skolem hull operator whose generators are augmented by all functions $\Psi^\alpha_\kappa \upharpoonright \alpha$.

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Michael Rathjen has shown in [9] that starting with $\Xi$, the first weakly compact cardinal, this procedure suffices to obtain a set of ordinals that suffices for an analysis of Kripke–Platek set theory augmented by the scheme of $\Pi_3$–reflection.
Already the iteration of a Skolem hull operator has the flavor of “elimination of ideal” objects. Though defined with the aid of large “ideal” ordinals the eventually obtained sets $\mathcal{H}^\alpha(X)$ (for finite $X$) are primitive recursively definable and the $<-$relation restricted to $\mathcal{H}^\alpha(X)$ is primitive recursively decidable. Therefore these sets are definable without reference to large cardinals and $\Psi_\Omega$ is actually collapsing below $\omega_1^{CK}$. 
With the aid of iterated Skolem hull operators we obtain the following main theorem.

**Theorem**

Let $F$ be a formula belonging to the complexity class of the ideal ordinal $\pi$. If all parameters of $F$ have stages less than $\pi$, then $\mathcal{H} \frac{\alpha}{\kappa+1} F^{L_\pi}$ implies $\mathcal{H} \omega^{\alpha+\kappa+\mu+1} \frac{\psi^{\omega^{\alpha+\kappa+\mu}}}{\psi^{\omega^{\alpha+\kappa+\mu}}} F^{L_\mu}$ for all $\mu \in \mathcal{M}_\pi^{\omega^{\alpha+\kappa}}$.

Since $\psi^{\omega^{\alpha+\kappa+\mu}}$ is always an ordinal below $\pi$ no (rules for) ideal ordinals $\geq \pi$ will occur in the derivation that is reduced according to Theorem 9.
Definition

A formal theory $T$ (in the language of set theory) has a model in ramified set theory at level $\pi$ if there is a Cantorian closed Skolem hull operator such that $H^\alpha_{\pi+n} F^{\pi}$ holds true for ordinals $\alpha < \varepsilon_{\pi+1}$ and $n < \omega$. 
Impredicative ordinal analysis

Definition

A formal theory $T$ (in the language of set theory) has a model in ramified set theory at level $\pi$ if there is a Cantorian closed Skolem hull operator such that $\mathcal{H} \frac{\alpha}{\pi+n} F^{L_\pi}$ holds true for ordinals $\alpha < \varepsilon_{\pi+1}$ and $n < \omega$.

So if $T$ is a theory that has a model in ramified set theory at $\pi$ and $T \models F^{L_\Omega}$ for a $\Pi_2$–sentence $F$ we obtain $\mathcal{H} \frac{\alpha}{\pi+1} F^{L_\Omega}$ by predicative cut elimination for an ordinal $\alpha < \varepsilon_{\pi+1}$. Since $\gamma := \Psi^\omega_{\Omega} \in M^\omega_{\Omega}$ we thus obtain $\mathcal{H}^{\xi} \frac{\beta}{\beta} F^{L_\gamma}$ for ordinals $\beta, \gamma < \Omega$. The derivation $\mathcal{H}^{\xi} \frac{\beta}{\beta} F^{L_\gamma}$ thus does not contain any ideal ordinals and thus also no rules for ideal ordinals.
Let us first cite a definition which is due to Andreas Weiermann (cf. [2]). Given a function
Φ: ℕ → ℕ and a set O of ordinals such that to every ordinal
α ∈ O there is a norm N(α) < ω. Then we can define a
function
ψ: O → ω by
ψ(α) := sup {ψ(β) + 1| β < α ∧ N(β) < Φ(N(α))} ∪ {0}.
Defining Φα(x) := ψ(ω·α + x) we obtain Φn(x) ≈ Φn(x) where
the latter means the familiar iteration of the function Φ.
Therefore Φα is closely connected to the more familiar fast
growing hierarchy Φα which is defined by

Φ0(x) = x,  Φα+1(x) = Φ(Φα(x)) and  Φλ(x) = Φλ[x](x)

for limit ordinals λ where λ[x] denotes the xth member of a
fundamental sequence for λ. Starting with a primitive recursive
function Φ satisfying some mild preconditions we obtain that
Φε0 designs an experiment for Peano arithmetic.
The ordinals in $\mathcal{H}(X)$ can be notated by finite strings built up from the generators of $\mathcal{H}$ and the parameters in $X$. This can be made unique such that every ordinal $\alpha \in \mathcal{H}(X)$ can be equipped with a norm function $N(\alpha)$ which is the length of its notation. Let $\Phi : \mathbb{N} \to \mathbb{N}$ be a primitive recursive function.

**Definition**

Let $X$ be a set of ordinals. Then we define recursively

$$\psi_{\mathcal{H}(X)}^\Phi(\alpha) := \sup \left\{ \psi_{\mathcal{H}(X)}^\Phi(\beta) + 1 \mid \beta \in \mathcal{H}(X) \cap \alpha \land N(\beta) < \Phi(N(\beta)) \right\} \cup \{0\}.$$ 

The function $\psi_{\mathcal{H}(X)}^\Phi$ therefore collapses $\alpha$ below $\omega$ and we define for regular cardinals $\kappa$ the function $\Phi_{\mathcal{H}(X)}^\kappa : \mathbb{N} \to \mathbb{N}$ by

$$\Phi_{\mathcal{H}(X)}^\kappa(\alpha) := \psi_{\mathcal{H}(X)}^\Phi(\omega \cdot \psi_\kappa^\alpha + x).$$
For a $\Pi^0_2$–analysis we have also to take care of the finite parameters. Therefore we define for a finite set $X$ of ordinals

$$|X| := \max(\{N(\alpha) + 1 \mid \alpha \in X\} \cup \{\bar{X} + 1\})$$

and define the fragmented iteration $f^\mathcal{H}_{\kappa,\alpha}(X)$ by

$$f^\mathcal{H}_{\kappa,\alpha}(X) := \{\xi \in \mathcal{H}^{\alpha}(X) \mid N(\xi) < \Phi^\mathcal{H}_{\kappa,\alpha}(X)(|X|)\}.$$
Then we define fragmented controlled derivations in complete analogy to operator controlled derivations in ramified set theory.

**Definition**

Let \( \frac{\alpha}{\rho} \Delta \) be a derivation in a semi–formal system. We say that a fragmented Skolem hull operator \( fH_{\kappa}^{\gamma} \) controls \( \frac{\alpha}{\rho} \Delta \), written as \( fH_{\kappa}^{\gamma} \frac{\alpha}{\rho} \Delta \), if \( \alpha \in fH_{\kappa}^{\gamma}(par(\Delta)) \) and for every inference

\[
\frac{\alpha_{i}}{\rho} \Delta_{i} \Rightarrow \frac{\alpha}{\rho} \Delta
\]

which is not an inference according to \((\wedge)\), we have \( par(\Delta_{i}) \cup \{|\Delta|_{i}\} \subseteq fH_{\kappa}^{\gamma}(par(\Delta)) \).
Let $\Phi^\alpha_k(X) := \Phi^H_{k,\alpha}(X)(|X|)$.

**Theorem**

(Witnessing Theorem) Assume that $fH^\gamma_{k,0} F^{L_\omega}$ holds true for a $\Sigma_1$–formula $(\exists x)F(x)$. Then there is a $t \in L_m$ such that $L_\omega \models F(t)$ and $m < \Phi_{k,\omega}^{\gamma + \omega \alpha + 1}(\text{par}(F))$. 
The main theorem (Theorem 9) modifies to

**Theorem**

Let $F$ be a formula belonging to the complexity class of the ideal ordinal $\pi$. If all parameters of $F$ have stages less than $\pi$ then $\vdash_{H}^{\gamma+1} \frac{\alpha}{\kappa+1} F^{L_\pi}$ implies $\vdash_{H}^{\gamma+\omega\alpha+\kappa+\mu+1} \frac{\Psi_{\pi}^{\gamma+\omega\alpha+\kappa+\mu}}{\Psi_{\pi}^{\gamma+\omega\alpha+\kappa+\mu}} F^{L_\mu}$ for all $\mu \in M_\pi^{\omega+\kappa}$.
Together with the Witnessing Theorem we thus obtain the main theorem.

**Theorem**

*Assume that a theory in the language of set theory has a model in ramified set theory at level $\pi$. Then the function $\Phi^{\varepsilon}_{\pi+1}$ designs an experiment for $T$.\*

The details of this approach are in [8]
To sum up I want to make the following statements:

- there are two types of “ideal objects”.
  - Abstract ideal objects which are those objects which need “impredicative abstract notions” in their definition.
  - Strict ideal statements which are all statements whose formulation exceeds $\Pi_2^0$.

- Ordinal analysis for impredicative theories includes – or better, is based on – elimination of abstract ideal objects for the cost of the introduction of infinite derivations.

- An elimination of all ideal notions in a theory $T$ means to design an experiment for $T$, which in turn needs to extend it´s ordinal analysis to a $\Pi_2^0$–analysis.
Thank you for your attention


———, *Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie*, *Forschungen zur Logik und Grundlegung der exakten Wissenschaften*, vol. 4 (1938), pp. 19–44.

———, *Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der rehen*


