Stability analysis of adaptive algorithms for blind source separation of convolutive mixtures

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Abstract

In this paper we present a stability analysis of two adaptive algorithms proposed in (Nguyen-Thi and Jutten, 1995) for blind source separation of convolutive mixtures. We demonstrate that for temporally white sources and causal mixing filters the first algorithm is not stable whereas the second is always stable provided that the step-size parameters are adequately chosen. Moreover, we show that the second algorithm exhibits isotropic convergence. © 1999 Elsevier Science B.V. All rights reserved.

Zusammenfassung

In diesem Artikel präsentieren wir eine Stabilitätsanalyse zweier adaptiver Algorithmen, die in (Nguyen-Thi und Jutten, 1995) zur blinden Quellenseparation von gefalteten Mischungen vorgeschlagen wurden. Wir demonstrieren, dass der erste Algorithmus für zeitlich weiße Quellen und kausale Mischer nicht stabil ist, während der zweite, mit adäquat gewählten Schrittweitenparametern, immer stabil ist. Des weiteren zeigen wir, dass der zweite Algorithmus eine isotropische Konvergenz liefert. © 1999 Elsevier Science B.V. All rights reserved.

Résumé

Nous présentons dans cet article une analyse de la stabilité des deux algorithmes adaptatifs proposés dans (Nguyen-Thi et Jutten, 1995) pour la séparation aveugle de sources pour des mélanges convolutifs. Nous démontrons que pour des sources temporellement blanches et des filtres de mélange causals le premier algorithme n’est pas stable alors que le second est toujours stable pourvu que les paramètres de modification soient choisis adéquatement. De plus, nous montrons que le second algorithme est caractérisé par une convergence isotropique. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Blind source separation; Convolutive mixtures; Multichannel equalization; Adaptive signal processing; High-order statistics

1. Introduction

Blind source separation consists in the retrieval of a set of original signals termed sources from observations of linear mixtures of them. This problem arises in a large number of applications in...
Nomenclature

\[ A(z) \] mixing system in the z-domain \((N \times N)\)
\[(I + C(z))^{-1} \] separating system \((N \times N)\)
\[ C_s(z) \] optimum matrix for the separation \((N \times N)\)
\[ D \] auxiliar Jacobian matrix \((N^2(L + 1) \times N^2(L + 1))\)
\[ J \] Jacobian matrix \((N^2(L + 1) \times N^2(L + 1))\)
\[ x[n] \] vector of sources \((N \times 1)\)
\[ y[n] \] vector of observations \((N \times 1)\)
\[ s[n] \] vector of outputs \((N \times 1)\)
\[ \theta \] vector rearrangement of \( C[n] \) \((N^2(L + 1) \times 1)\)
\[ \Delta \] vector rearrangement of the minus update \((N^2(L + 1) \times 1)\)
\[ L \] length minus one of the separation filters
\[ c_{ij}[k] \] \((i, j)\) element of matrix \( C[k] \)
\[ w_s[n] \] auxiliar signal constructed from the zero lag mixing coefficients
\[ \mu_j \] step size element
\[ \eta \] parameter of the step size function
\[ \rho(\cdot, \cdot), \Phi \] performance index of the separation
\[ \delta_{ij} \] Kronecker function
\[ \delta[n] \] Dirac delta function
\[ [C(z)]y[n] \] abbreviated notation for the convolution of \( C[n] \) and \( y[n] \)
\[ K_{x_i} \] fourth-order cumulant (kurtosis) of the \( i \)th source
\[ K_{x_i} \] fourth-order cumulant (kurtosis) of the \( i \)th output
\[ \text{Cum}(\alpha, \ldots, \beta) \] cross-cumulant notation
\[ \text{Cum}_{3,1}(\alpha, \beta) \] abbreviated notation for \( \text{Cum}(\alpha, \alpha, \alpha, \beta) \)
\[ \text{Cum}_{1,3}(\alpha, \beta) \] abbreviated notation for \( \text{Cum}(\alpha, \beta, \beta, \beta) \)
\[ \text{Cum}_{1,2,1}(\alpha, \beta, \gamma) \] abbreviated notation for \( \text{Cum}(\alpha, \beta, \beta, \gamma) \)

Signal processing and can be solved assuming only that the sources are statistically independent non-Gaussian distributed and that the mixing system is invertible.

Let \( x[n] = [x_1[n], x_2[n], \ldots, x_N[n]]^T \) be a vector of \( N \) unknown signals termed sources that propagate along an open medium and let \( y[n] = [y_1[n], y_2[n], \ldots, y_N[n]]^T \) be the vector of \( N \) observed signals provided by an array of sensors that spatially sample this medium. Observations are typically related to the sources through

\[ y[n] = \sum_{k=0}^{\infty} A[k]x[n - k] = [A(z)]x[n], \quad (1) \]

where \( A(z) = \sum_{k=0}^{\infty} A[k]z^{-k} \) represents the causal transfer function of the mixing system. The objective in blind source separation is to recover the sources by a linear filtering of the observations

\[ s[n] = \sum_{k=0}^{L} B[k]y[n - k] = [B(z)]y[n], \quad (2) \]

where \( B(z) = \sum_{k=0}^{L} B[k]z^{-k} \) is the transfer function of a matrix of \( L + 1 \) taps FIR filters that represents the separating system.

Following the work of Nguyen and Jutten [5], we will assume that the separating system has a recursive structure so that the outputs are

\[ s[n] = y[n] - \sum_{k=0}^{L} C[k]s[n - k], \quad (3) \]

where \( C[k] \) is an impulse response matrix of \( L + 1 \) taps FIR filters where the diagonal elements are set to zero, i.e., \( c_{ii}[k] = 0, \forall i,k \). Thus, in the z-domain

\[ B(z) = (I + C(z))^{-1}, \quad (4) \]
where $C(z)$ is the transfer function of the adaptive part in the separating system.

Two algorithms based on fourth-order cross-cumulants are proposed in [5] to select $C(z)$ for a $2 \times 2$ separating system.

Algorithm I:

\[
cl_{[n]}^{[n+1]}[k] = cl_{[n]}^{[n]}[k] - \mu_j \text{Cum}_{31}(s_l[n], s_r[n-k]),
\]

(5)

Algorithm II:

\[
cl_{[n]}^{[n+1]}[k] = cl_{[n]}^{[n]}[k] - \mu_j \text{Cum}_{13}(s_l[n], s_r[n-k])
\]

(6)

for all $i,j \neq 1, 2; k = 0, \ldots, L$. It is demonstrated in [5] that for a $2 \times 2$ mixing system the points where signals are perfectly recovered are stationary points of both algorithms. However, nothing is said about the algorithm stability at these points.

In this paper we will present a stability analysis of the above algorithms at the separating solutions assuming temporally independent and identically distributed (i.i.d.) sources. We will show that algorithm I is not stable and that algorithm II is stable provided that the sign and the magnitude of the step size is adequately chosen.

2. Stability analysis

To simplify notation, let us group the separating matrix coefficients in the vector $\theta$ as

\[
\theta = [c_{11}[0], c_{21}[0], \ldots, c_{12}[k],
\]

\[
c_{21}[k], \ldots, c_{12}[L], c_{21}[L]^T
\]

(7)

Algorithms I and II can therefore be rewritten as

\[
\theta^{[n+1]} = \theta^{[n]} - A^{[n]},
\]

(8)

where $A^{[n]} = [A^{1[n]}, \ldots, A^{(n)}]$ and $A^{(n)}$ is defined as

\[
A^{(n)} = [\mu_2 \text{Cum}_{31}(s_1[n], s_2[n-k]), \\
\mu_1 \text{Cum}_{31}(s_2[n], s_1[n-k])]^T
\]

(9)

for algorithm I and

\[
A^{(n)} = [\mu_2 \text{Cum}_{13}(s_1[n], s_2[n-k]), \\
\mu_1 \text{Cum}_{13}(s_2[n], s_1[n-k])]^T
\]

(10)

for algorithm II.

The stationary points of the recursion (8) are the points where $A$ vanishes. It is demonstrated in [5] that $A$ vanishes for both algorithms I and II at the optimum matrix $C_*(z)$ where separation is achieved, i.e., when $s[n] = x[n]$. It is well known [4] that the algorithms will converge towards them if the modulus of all the eigenvalues of the adaptation Jacobian matrix is strictly less than one.

The Jacobian of the adaptation rule (8) is

\[
J = \frac{\partial \theta^{[n+1]}}{\partial \theta^{[n]}} = I - D,
\]

(11)

where $I$ is the identity matrix and $D = \partial A^{(n)}/\partial \theta^{[n]}$. Taking into account the cumulants properties [6] and the hypothesis on the sources and the mixture system, the elements of $D$ for algorithm I are given by (see Appendix A)

\[
\mu_j \frac{\partial \text{Cum}_{31}(s_l[n], s_r[n-k])}{\partial c_{mn}[m]} \bigg|_{s[n] = x[n]} = -\delta_{i,j} \delta_{m,k} w_*[k+m] K_{x},
\]

(12)

for $i,j,r,s ; i \neq j, r, s = 1, 2; k, m = 0, \ldots, L$,

where $K_{x} = \text{Cum}_d(x_1[n])$ is the kurtosis of the $i$th-source and $w_*[n]$ is the inverse z-transform of

\[
W_*(z) = \frac{1}{1 - A_{12}(z)A_{21}(z)}.
\]

(13)

Provided that $a_{ij}[n]$ are causal filters for $i,j \neq 1, 2$, $w_*[n]$ is also a causal filter ($w_*[n] = 0, \forall n < 0$). Also, the causality of $w_*[n]$ allows us to apply the initial value theorem [7] to obtain

\[
w_*[0] = \lim_{z \to \infty} W_*(z) = \frac{1}{1 - a_{12}[0]a_{21}[0]},
\]

(14)

where $1 - a_{12}[0]a_{21}[0] \neq 0$ as a consequence of the causality of $w_*[n]$ (see [2] for more details).

Substituting (12) in (11) yields to a Jacobian matrix $J = [J_{pq}]$ formed by $2 \times 2$ block matrices. The diagonal blocks are given by

\[
J_{pp} = \begin{pmatrix}
1 & \mu_2 w_*[2p] K_{x} \\
\mu_1 w_*[2p] K_{x} & 1
\end{pmatrix}
\]

(15)
for \( p = 0, \ldots, L \), whereas the off-diagonal blocks are given by

\[
J_{pq} = \begin{pmatrix}
0 & \mu_2 w_*[p + q] K_x \\
\mu_1 w_*[p + q] K_x & 0
\end{pmatrix}
\] (16)

for \( p, q = 0, \ldots, L \). The eigenvalues of \( J \) are not easy to find in general. However, when considering instantaneous mixtures \( J \) reduces to

\[
J = \begin{pmatrix}
1 & \mu_2 w_*[0] K_x & 0 & \ldots & 0 \\
\mu_1 w_*[0] K_x & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]

and its characteristic polynomial is

\[
P(\lambda) = (1 - \lambda)^2((1 - \lambda)^2 - \mu_1 \mu_2 (w_*[0])^2 K_x K_x^*)
\] (17)

Calculating the roots of \( P(\lambda) \) we obtain that the eigenvalues of \( J \) are \( \lambda = 1 \) with multiplicity 2L and

\[
\lambda = 1 \pm w_*[0] \sqrt{\mu_1 \mu_2 K_x K_x^*}
\]

with multiplicity 1. Thus, the eigenvalue of the Jacobian with maximum modulus, \( \lambda_{\text{max}} = 1 + |w_*[0]| \sqrt{\mu_1 \mu_2 K_x K_x^*} \), is always outside the unit circle if the kurtosis of the sources are non-zero. As a consequence, algorithm I will be always unstable for instantaneous mixtures.

However, algorithm II is stable for the general convolutive case, as will be shown in the sequel. In Appendix B the elements of \( D = \partial D^{(n)}/\partial \theta^{(n)} \) are calculated for algorithm II. They are given by

\[
\mu_j \frac{\partial \text{Cum}_{13}(s[n], s[j][n - k])}{\partial c_x[n]}_{x[n]=s[n]}
= - \delta_{ij} \delta_{jk} w_*[k - m] K_x^*
\]

for \( i, j, k, s_1, s_2, s_3 = 1, 2; \ k, m = 0, \ldots, L \). (19)

Taking into account the causality of \( w_*[n] \) and substituting (19) in (11) results in a lower triangular Jacobian matrix whose eigenvalues are

\[
\lambda_j = 1 + \mu_j w_*[0] K_x \\
= 1 + \mu_j \frac{K_{s_j}}{1 - a_{12}[0] a_{21}[0]} \quad \text{for} \ j = 1, 2,
\] (20)

each one with multiplicity \( L + 1 \). Therefore, local convergence can be obtained if we choose the step size \( \mu_j \) in order that the modulus of the eigenvalues be bounded by 1. Moreover, isotropic convergence is obtained when all the diagonal elements of \( D \) are identical. Both properties, asymptotic stability and isotropic convergence, are guaranteed whenever the step size is selected according to

\[
\mu_j = \frac{1 - \eta}{a_{12}[0] a_{21}[0]} \quad \text{for} \ j = 1, 2,
\] (21)

where \( 0 < \eta < 2 \). Note that when \( 1 > a_{12}[0] a_{21}[0] \) the sign of the step size should be opposite to the corresponding source kurtosis sign for the algorithm to converge. The fastest asymptotic convergence is obtained when \( \eta = 1 \) since the eigenvalues of the Jacobian matrix are zero in this case. The kurtosis of the sources \( K_x \) can be estimated from the kurtosis of the outputs \( K_{s_j}^{(n)} \) using an efficient recursive procedure such as that described in [3], whereas the mixing coefficients for zero lag \( a_{ij}[0] \) can be replaced at each iteration by their estimates at that time, i.e., \( c_{ij}[n] \). Thus, we propose the following step-size:

\[
\mu_j^{(n)} = \frac{1 - c_{12}[0] c_{21}[0]}{K_{s_j}^{(n)}} \quad \text{for} \ j = 1, 2.
\] (22)

In some cases, specifically when our estimates of the kurtosis are not much accurate, it is desirable to avoid that the denominator vanishes. Therefore, we can use a more robust version of the step size given by

\[
\mu_j^{(n)} = - \eta \text{sign}(K_{s_j}) \frac{1 - c_{12}[0] c_{21}[0]}{a + |K_{s_j}^{(n)}|} \quad \text{for} \ j = 1, 2,
\] (23)

where \( a \) is some small positive constant and \( 0 < \eta \leq 1 \).

3. Simulations

Computer simulations were carried out to corroborate the theoretical results obtained in the previous section. In a first computer experiment we
considered an instantaneous mixture of two temporally i.i.d. binary sources \( (K_{x_i} = K_{x_2} = -2) \) and the mixing matrix
\[
A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.
\] (24)

\[
A(z) = \frac{1}{0.2 - 0.2z^{-1} - 0.25z^{-2} - 0.2z^{-4}} \begin{bmatrix} 0.2 + 0.2z^{-1} + 0.2z^{-2} - 0.25z^{-3} + 0.25z^{-4} \\ 1 \end{bmatrix}.
\] (25)

Figs. 1 and 2 plot the mean vector field [1] of both algorithms. Note that since this quantity does not have sense in the whole space we restrict the plots to the neighborhood of the optimum separation point \([c_{12}, c_{21}] = [0.5, 0.5] \). This point is denoted in the graphics by the symbol "o". It is easy to check in Fig. 1 that algorithm I will not be stable at the separation point whereas Fig. 2 shows that algorithm II will be locally stable at separation and with isotropic convergence.

In a second computer experiment we considered a convolutive mixture of two binary sources and the following mixing channel:

\[
C_4(z) = \begin{bmatrix} 0 & A_{12}(z) \\ A_{21}(z) & 0 \end{bmatrix}.
\] (26)

Separation occurs when

Thus, the optimum separating matrix is given by

\[
B_*(z) = (I + C_*(z))^{-1} = \frac{1}{\det(A(z))} (I - C_*(z)).
\] (27)

---

**Fig. 1.** Dynamic behaviour of algorithm I. The separating solution "o" is an unstable point.
Note that the denominator det(\(A(z)\)) determines the stability of the optimum separation system. In the present case, \(\text{det}(A(z)) = 1.04 - 0.08z^{-1} - 0.05z^{-2} + 0.14z^{-3} - 0.09z^{-4} + 0.0275z^{-5} + 0.1025z^{-6} - 0.05z^{-7} + 0.05z^{-8}\) and the maximum modulus root of this polynomial \((z = -0.72 \pm 0.28i)\) lies inside the unit circle thus indicating that the optimum separation system is stable.

Algorithm implementations were carried out in a batch mode where the cumulants were estimated from data vectors of 1000 samples as follows:

\[
\overline{\text{Cum}}_{13}(s_i[n], s_j[n - k]) = E_{av}[s_i[n]s_j[n - k]]^3 - 3s_i[n]s_j[n - k]E_{av}[s_j[n - k]s_j[n - k]],
\]

(28)

\(E_{av}[\cdot]\) denotes the temporal average estimator. The algorithm step size was selected according to (22) with the optimal factor \(\eta = 1\). To evaluate the algorithms performance we chose the Euclidean distance \(\rho(A(z), I + C(z))\) between the temporal coefficients of the true mixing system and the estimated mixing system. Fig. 3 plots the time evolution of this performance index for algorithms I and II. It is apparent that algorithm I diverges towards unstable separating systems (see dashed-dotted line) while algorithm II converges in two iterations (continuous line). This fast convergence remarks the validity of the proposed step size.

We carried out a third computer experiment to illustrate the behaviour of the algorithms with a little harder mixture. We replaced the mixing
matrix $A(z)$ by
\[
A_1(z) = \begin{bmatrix}
1 & 2A_{12}(z) \\
2A_{21}(z) & 1
\end{bmatrix},
\] (29)
whose determinant is $\text{det}(A_1(z)) = 1.16 - 0.32z^{-1} - 0.20z^{-2} + 0.56z^{-3} - 0.36z^{-4} + 0.11z^{-5} + 0.41z^{-6} - 0.20z^{-7} + 0.20z^{-8}$. Note that the optimum separation system $B_*(z)$ is still stable since the modulus of the outermost root of $\text{det}(A_1(z))$ is slightly less than the unity (in particular, its value is $|z| = 0.92$).

Fig. 5 plots the time evolution of the Euclidean distance performance index for the new mixture. Again, algorithm II converges in a few iterations (see the dashed line) while algorithm I diverges (dashed-dotted line).

An alternative performance index that can be used to evaluate algorithm II is
\[
\Phi = \sum_{i,j,l,u} \sum_{k=0}^L |\text{Cum}_{13}(s_i[n],s_j[n-k])|.
\] (30)

This index vanishes at convergence. Fig. 4 plots the evolution of $\Phi$ versus iterations for the mixing systems $A(z)$ and $A_1(z)$. It is apparent from this figure that algorithm II quickly exploits all the information present in the employed statistics since they are driven to zero.

Finally, we considered an on-line implementation of the algorithms. Now the cumulants $\text{Cum}_{13}(s_i[n],s_j[n-k])$ are replaced by its stochastic approximation
\[
\widehat{\text{Cum}}_{13}(s_i[n],s_j[n-k]) = s_i[n](s_j[n-k] + 3s_j[n]s_j[n-k]) - 3s_i[n]s_j[n-k]s_j[n-k]
\] (31)
where $\sigma_j^2$ is the power of the $j$th-signal that is estimated by using the following recursion:
\[
\sigma_j^{2(n)} = (1 - \alpha)\sigma_j^{2(n-1)} + \alpha s_j^2[n]
\] (32)
with $\alpha = 0.01$. The step size for the algorithm was chosen as in (22) using $\eta = 0.001$. Fig. 5 plots the
Euclidean distance index versus iterations when $A_1(z)$ is the mixing system. It is clearly seen that algorithm I diverges while algorithm II converges to the optimum separation filter. Fig. 6 plots the outputs versus iterations for algorithm II.

4. Conclusions

This paper presents the stability analysis of two algorithms for blind source separation of convolutive mixtures proposed in [5]. The analysis consists in examining the eigenvalues of the Jacobian matrix associated to the algorithms. We have demonstrated that for causal mixtures and temporally white sources algorithm II is stable whereas algorithm I is not. Moreover, we demonstrate that algorithm II exhibits isotropic convergence when the step-size parameters are adequately chosen.

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Appendix A. Calculation of matrix $D$ for algorithm I

Taking into account the cumulants properties [6] it is easy to show that
\[
\frac{\partial \text{Cum}_{31}(s_i[n], s_j[n-k])}{\partial c_{re}[m]}
= \frac{\partial \text{Cum}(s_i[n], s_i[n], s_j[n], s_j[n-k])}{\partial c_{re}[m]}
\]
Fig. 5. Convergence of the separation algorithms: dashed line for algorithm I, continuous line for algorithm II.

\[
\begin{align*}
= \text{Cum} \left( s_i[n], s_i[n], s_i[n], \frac{\partial s_i[n-k]}{\partial c_{r_i}[m]} \right) \\
+ 3 \text{Cum} \left( s_i[n], s_i[n], s_j[n-k], \frac{\partial s_i[n]}{\partial c_{r_i}[m]} \right)
\end{align*}
\]

(A.1)

for \( i, j \neq i, j = 1, 2; r, s | r \neq s | = 1, 2; k, m = 0, \ldots, L \).

From Eq. (3) we can see that

\[
s_i[n] = y_i[n] - \sum_{k=0}^{L} c_{ij}[k] s_j[n-k]
\]

(A.2)

and, as a consequence,

\[
\frac{\partial s_i[n]}{\partial c_{ij}[m]} = - s_i[n-m] - [C_{ij}(z)] \frac{\partial s_i[n]}{\partial c_{ij}[m]},
\]

(A.3)

\[
\frac{\partial s_j[n]}{\partial c_{ij}[m]} = - [C_{ji}(z)] \frac{\partial s_i[n]}{\partial c_{ij}[m]},
\]

(A.4)

Therefore [2]

\[
\frac{\partial s_i[n]}{\partial c_{ij}[m]} = - [W(z)] s_j[n-m],
\]

(A.5)

\[
\frac{\partial s_j[n]}{\partial c_{ij}[m]} = - [W(z)C_{ji}(z)] s_i[n-m]
\]

(A.6)

for \( i, j \neq i, j = 1, 2; k, m = 0, \ldots, L \), and where \( W(z) = (1 - C_{ij}(z)C_{ji}(z))^{-1} \).

Thus,

\[
\frac{\partial \text{Cum}_{31}(s_i[n], s_j[n-k])}{\partial c_{ij}[m]}
\]

\[
= - \text{Cum}_{31}(s_i[n], [W(z)C_{ji}(z)] s_j[n-k-m])
\]

\[
- 3 \text{Cum}_{211}(s_i[n], s_j[n-k], [W(z)] s_j[n-m])
\]

(A.7)
and

\[
\frac{\partial \text{Cum}_3(s_i[n], s_j[n-k])}{\partial c_{ji}[m]}
= - \text{Cum}_3(s_i[n], [W(z)]s_i[n-k-m])
- 3\text{Cum}_2(s_i[n], s_j[n-k], [W(z)C_{i,j}(z)]s_i[n-m]).
\]  

(A.8)

One of the cumulants properties is that they vanish when their different arguments include at least two independent variables. At the separation point \(s[n] = x[n]\) most of the terms in Eqs. (A.7) and (A.8) will vanish since they involve two independent signals and

\[
\frac{\partial \text{Cum}_3(s_i[n], s_j[n-k])}{\partial c_{ij}[m]} \bigg|_{s[n]=x[n]} = 0,
\]  

(A.9)

\[
\frac{\partial \text{Cum}_3(s_i[n], s_j[n-k])}{\partial c_{ji}[m]} \bigg|_{s[n]=x[n]} = 0,
\]  

(A.10)

where \(w_\ast[n]\) is the inverse z transform of \(W_\ast(z) = (1 - A_j(z)A_j(z))^{-1}\).

Taking into account that for temporally i.i.d. sources \(\text{Cum}_3(x_i[n], x_i[n-p]) = \delta[p] K_{x_i}\), where \(K_{x_i} = \text{Cum}_4(x_i[n])\) is the kurtosis of the ith-source,
\[
\frac{\partial \text{Cum}_3(s_i[n], s_j[n - k])}{\partial c_{il}[m]} \bigg|_{s[n] = x[n]} = -w_\ast[k + m] K_x.
\]
\[
\text{(A.12)}
\]

Therefore, we can finally write that the elements of \( D \) are proportional to
\[
\frac{\partial \text{Cum}_3(s_i[n], s_j[n - k])}{\partial c_{il}[m]} \bigg|_{s[n] = x[n]} = -\delta_{i0}\delta_{j0}w_\ast[k + m] K_x,
\]
\[
\text{(A.13)}
\]

for \( i, j \neq 1, 2; r, s \neq 1, 2; k, m = 0, \ldots, L \).

Appendix B. Calculation of matrix \( D \) for algorithm II

Following the same procedure as in Appendix A we find that the cross-cumulants involved in the calculation of the elements of \( D \) at separation for algorithm II are given by
\[
\frac{\partial \text{Cum}_3(s_i[n], s_j[n - k])}{\partial c_{il}[m]} \bigg|_{s[n] = x[n]} = -\text{Cum}_3(W_\ast(z)x_i[n - m], x_j[n - k])
\]
\[
= - \sum_{l = -\infty}^{\infty} w[l] \text{Cum}_3(x_i[n - m - l], x_j[n - k]),
\]
\[
\text{(B.1)}
\]
\[
\frac{\partial \text{Cum}_3(s_i[n], s_j[n - k])}{\partial c_{il}[m]} \bigg|_{s[n] = x[n]} = 0
\]
\[
\text{(B.2)}
\]

for \( i, j \neq 1, 2; r, s = 1, 2; k, m = 0, \ldots, L \). Again, the temporally i.i.d. assumption for the sources leads to
\[
\frac{\partial \text{Cum}_3(s_i[n], s_j[n - k])}{\partial c_{il}[m]} \bigg|_{s[n] = x[n]} = -\delta_{i0}\delta_{j0}w_\ast[k - m] K_x,
\]
\[
\text{(B.3)}
\]

for \( i, j \neq 1, 2; r, s = 1, 2; k, m = 0, \ldots, L \).

References